Unit Wise Question Bank With Solutions

Mathematics II (BS103MT), 2019-20

For

B.E. I year (AICTE) O.U.

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SAQ's:- **Unit I, Matrices**

J-15

(1) Find all values of λ for which rank of the matrix $\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$ is equal to 3.

D-16

(2) Determine the value of k for which the matrix
$$A = \begin{bmatrix} 6 & 3 & 5 & 9 \\ 5 & 2 & 3 & 6 \\ 3 & 1 & 2 & k \end{bmatrix}$$ is of rank 3.

D-19

(3) Reduce the following matrix to row echelon form and find its rank $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 2 \\ 5 & -5 & 11 \end{bmatrix}$

M-19

(4) Examine the linear independence of vectors
(1, 1, 0, 1); (1, 1, 1, 1); (-1, 1, 1, 1); (1, 0, 0, 1).

D-16

(5) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ then find its spectrum and spectral radius.

D-16

(6) Write any two properties of eigen values.

J-15

(7) If the sum of eigen values of matrix $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & k & 2 \\ -1 & 2 & 2k \end{bmatrix}$ is 10, then find k.

D-17

(8) Find the sum & product of eigen values of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

J-17

(9) Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector of $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ corresponding to the eigen value 5.

M-19

(10) (a) State Cayley-Hamilton Theorem.

(b) Verify Cayley-Hamilton Theorem for $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

(11) D-15 (a) Define linear transformation.

M-19 (b) Define orthogonal transformation.

D-15 (12) Obtain the symmetric matrix A for the quadratic form $Q = x^2 + 2y^2 + 3z^2 + 4xy + 8yz + 6xz$.

M-19 (13) Reduce the Q.F $3x^2 + 5y^2 + 3z^2 - 2yx + 2zx - 2xy$ to canonical form.

LAQs:-

(1) J-17 D-17 Test the consistency of the equations $x+y+z=6$, $x-y+2z=5$, $3x+y+z=8$ and $2x-2y+3z=7$ and hence solve.

D-16 (2) Find the values of λ so that the equations $x+y+z=1$, $2x+y+4z=\lambda$, $4x+y+10z=\lambda^2$ have a solution and solve them completly in each case.

D-15 (3) Determine the values of k for which the system of equations $x-ky+z=0$, $kx+3y-kz=0$ $3x+y-z=0$ has
(i) only zero solution (ii) non-zero solution

J-17 (4) Find the values of a & b such that the equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+az=b$ have
(i) no solution (ii) unique solution
(iii) infinite solutions.

(5) J-17, D-17, D-19 Find all the eigen values & corresponding eigen vectors of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ (10 Marks)

M-19

(6) Find the eigen values & eigen vectors of $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(7) J-15, D-15, A-16 If $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ then find A^{-1} by using Cayley-Hamilton Theorem.

J-16, J-17

(8) Find the characteristic equation of $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and hence find the matrix $A^8 - 5A^7 - A^6 - 5A^5 - A^4 + 6A^2 + I$.

D-16

(9) If $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Find the eigen values of $3A^5 - A^4 + A^2 + 3I - A^{-1}$.

J-15, D-19

(10) Reduce the Quadratic form $Q = 2(xy + yz + zx)$ to Canonical form by orthogonal transformation and find its nature (10 marks)

(11) D-15, A-16, D-16, J-17, M-19. Reduce the Q.F $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1$ to canonical form through orthogonal transf and find rank, index & signature.

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Solutions

SAQs:

1. Find all values of λ for which rank of matrix $\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$ is equal to 3.

Given: $A = \begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$ & $\rho(A) = 3$.

det of $A = 0$.

$$\lambda \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 11 & -6 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 & 0 \\ 0 & \lambda & -1 \\ -6 & -6 & 1 \end{vmatrix} = 0$$

$\lambda[\lambda(\lambda-6)+11] + 1[-6] = 0 \Rightarrow \lambda(\lambda^2 - 6\lambda + 11) - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1, 2, 3.$

2. Determine the value of k for which the matrix $A = \begin{bmatrix} 6 & 3 & 5 & 9 \\ 5 & 2 & 3 & 6 \\ 3 & 1 & 2 & k \end{bmatrix}$ is of rank 3.

Given: $\rho(A) = 3$

3rd order minor of A should be non-zero

i.e., $\begin{vmatrix} 3 & 5 & 9 \\ 2 & 3 & 6 \\ 1 & 2 & k \end{vmatrix} \neq 0$

$3(3k-12) - 5(2k-6) + 9(4-3) \neq 0$

$9k - 36 - 10k + 30 + 36 - 27 \neq 0$

$-k + 3 \neq 0$

$k \neq 3.$

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3. Reduce the following matrix to row-echelon form and find its rank $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 2 \\ 5 & -5 & 11 \end{bmatrix}$ **(6)**

Given: $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 2 \\ 5 & -5 & 11 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -4 \\ 5 & -5 & 11 \end{bmatrix}$

$R_3 \to R_3 - 5R_1$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -4 \\ 0 & 5 & -4 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

This is in row-echelon form.
As no. of non-zero rows $= 2$, $\rho(A) = 2$.

4. Examine linear independence of vectors.
(1,1,0,1) ; (1,1,1,1) ; (-1,1,1,1); (1,0,0,1)

Let $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1 \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 + R_1 \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$R_4 \to R_4 - R_1 \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3 \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

$R_4 \to 2R_4 + R_2 \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

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$R_4 \to R_4 - R_3 \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

This is in row-echelon form, $\rho(A) = 4$

Since, $\rho(A)$ is equal to no. of vectors (4), the given matrix A is LI.

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5. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ then find its spectrum & spectral radius.

Char eqn is $\lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$.

$tr(A) = 3$,

$A_{11} = -3$, $A_{22} = 3$, $A_{33} = -1$ then $A_{11} + A_{22} + A_{33} = -1$,

$det(A) = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1(-3) - 2(0) - (0) = -3$

$\therefore$ char eqn is $\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & -1 & 3 \\ & 0 & 1 & -2 & -3 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda^2 - 3\lambda + \lambda - 3 = 0$

$\lambda(\lambda - 3) + (\lambda - 3) = 0$

$(\lambda + 1)(\lambda - 3) = 0 \Rightarrow \lambda = 3, -1$.

$\therefore$ Eigen values are: $\lambda = 1, -1, 3$.

Spectrum $= \{-1, 1, 3\}$

Spectral radius $= |3| = 3$.

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6. Write any two properties of Eigen values:

<u>Properties of Eigen Values</u>: Let λ be eigen value, x be eigen vector corresponding to λ.

i) $k\lambda$ is Eigen value of kA

We know, $Ax = \lambda x$

$(kA)x = (k\lambda)x$

ii) $\lambda - k$ is Eigen value of $A - kI$.

We know, $Ax = \lambda x$

$Ax - kx = \lambda x - kx \Rightarrow (A - kI)x = (\lambda - k)x$.

7. If sum of eigen values of matrix $A=\begin{bmatrix}1&4&5\\0&k&2\\-1&2&2k\end{bmatrix}$ is 10, then find k.

(8)

Given: $A=\begin{bmatrix}1&4&5\\0&k&2\\-1&2&2k\end{bmatrix}$

Characteristic eqn is $\lambda^3-tr(A)\lambda^2+(A_{11}+A_{22}+A_{33})\lambda-|A|=0$.

$tr(A)=3k+1$

$A_{11}=2k^2-4$, $A_{22}=2k+5$, $A_{33}=k$

$A_{11}+A_{22}+A_{33}=2k^2+2k+k-4+5=2k^2+3k+1$

$|A|=\begin{vmatrix}1&4&5\\0&k&2\\-1&2&2k\end{vmatrix}=2k^2-4-4(2)+5(k)$
$=2k^2-4-8+5k$
$=2k^2+5k-12$

$\Rightarrow \lambda^3-(3k+1)\lambda^2+(2k^2+3k+1)\lambda-(2k^2+5k-12)=0$.

This is in the form of $ax^3+bx^2+cx+d=0$.

Sum of roots $=\frac{-b}{a}=+(3k+1)$

Given that sum of eigen values $=10$.

$1+3k=10 \Rightarrow 3k=9 \Rightarrow k=3$

8. Find sum & product of eigen values of $\begin{bmatrix}1&0&0\\0&3&-1\\0&-1&3\end{bmatrix}$.

Given: $A=\begin{bmatrix}1&0&0\\0&3&-1\\0&-1&3\end{bmatrix}$

Char eqn is $\lambda^3-tr(A)\lambda^2+(A_{11}+A_{22}+A_{33})\lambda-|A|=0$.

$tr(A)=7$

$A_{11}+A_{22}+A_{33}=(9-1)+3+3=14$

$|A|=1(9-1)=8$.

$\Rightarrow \lambda^3-7\lambda^2+14\lambda-8=0$.

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 14 & -8\\ & 0 & 1 & -6 & 8\\ \hline & 1 & -6 & 8 & 0\end{array}$$

$\Rightarrow \lambda^2-6\lambda+8=0$

$\lambda^2-4\lambda-2\lambda+8=0$

$\lambda(\lambda-4)-2(\lambda-4)=0$

$(\lambda-2)(\lambda-4)=0$

$\lambda=2,4$

$\therefore$ Eigen values: 1, 2, 4

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Sum of eigen values $= 1+2+4 = 7$
Product of eigen values $= 1\times 2\times 4 = 8.$

9. Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector of $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ corresponding to eigen value 5.

To find Eigen vector corresponding to eigen value 5,

Take $Ax = 5x$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$(A - 5I)x = 0$$

$$\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4x_1 + 4x_2 = 0 \Rightarrow x_1 = x_2.$

$3x_1 - 3x_2 = 0 \Rightarrow x_1 = x_2$

Let $x_1 = x_2 = \alpha$

Eigen vector is $\begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

10. a) State Cayley - Hamilton Theorem:

CAYLEY- HAMILTON THEOREM: Every square matrix A satisfies its own characteristic equation i.e.,
if $|A - \lambda I| = 0$ (or) $\lambda^n - C_1\lambda^{n-1} + C_2\lambda^{n-2} + \cdots (-1)^{n-1}C_{n-1}\lambda + (-1)^n C_n = 0$
then $A^n - C_1A^{n-1} + C_2A^{n-2} + \cdots\cdots (-1)^{n-1}C_{n-1}A + (-1)^n C_n = 0.$

b) Verify Cayley - Hamilton theorem for $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Given: $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Char equ is $\lambda^2 - tr(A)\lambda + |A| = 0.$

$$\lambda^2 - 4\lambda - 5 = 0 \qquad \because |A| = 3-8 = -5.$$

$$\lambda^2 - 5\lambda + \lambda - 5 = 0$$

$$\lambda(\lambda - 5) + (\lambda - 5) = 0$$

$$(\lambda+1)(\lambda-5) = 0$$

$$\lambda = -1, 5$$

From CHT, $A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+4 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-9 & 16-16 \\ 8-8 & 10-10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ CHT is verified.

11. a) Define linear transformation:

<u>LINEAR TRANSFORMATION</u>: Let $P(x,y)$ be a point in $\overline{XY}$ plane which is transformed to a point $P'(x',y')$ in $\overline{X'Y'}$ plane by following relations-

$x' = a_1x + a_2y$, $y' = b_1x + b_2y$ (or)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{(or)}$$

$X = AY$ where $X = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, $Y = \begin{bmatrix} x \\ y \end{bmatrix}$

Such a transformation is called linear transformation in two variables or 2D.

b) Define orthogonal transformation.

<u>ORTHOGONAL TRANSFORMATION</u>: A linear transformation $X = AY$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is said to be orthogonal if matrix A is orthogonal i.e., if $A^TA = I$.

12. Obtain symmetric matrix A for quadratic form $Q = x^2 + 2y^2 + 3z^2 + 4xy + 8yz + 6xz$.

Compare with $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxz$ -

$a=1,\ b=2,\ c=3,\ h=2,\ f=5/2,\ g=3$

Symmetric matrix is $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$

SAQ

13. Reduce the QF $3x^2+5y^2+3z^2-2yz+2xz-2xy$ to canonical form:

Given: $3x^2+5y^2+3z^2-2yx+2xz-2xy$

Compare with $ax^2+by^2+cz^2+2hxy+2fyz+2gxz$

$a=3,\ b=5,\ c=3,\ h=-1,\ f=-1,\ g=+1$

Symmetric matrix is $A=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Char eqn is: $\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$

$tr(A)=11\ ;\ A_{11}+A_{22}+A_{33} = 14+8+14 = 36.$

$|A| = 3(15-1)+(-3+1)+(1-5) = 42-2-4 = 36$

$\Rightarrow \lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$

$\lambda = 6, 3, 2.$

Canonical form: $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

$6y_1^2 + 3y_2^2 + 2y_3^2.$

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LAQs:

1. Test the consistency of eq^ns $x+y+z=6$, $x-y+2z=5$, $3x+y+z=8$ & $2x-2y+3z=7$ & hence solve.

$$A=\begin{bmatrix}1&1&1\\1&-1&2\\3&1&1\\2&-2&3\end{bmatrix},\quad b=\begin{bmatrix}6\\5\\8\\7\end{bmatrix}$$

$$AX=b \Rightarrow \begin{bmatrix}1&1&1\\1&-1&2\\3&1&1\\2&-2&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\5\\8\\7\end{bmatrix}$$

Augmented matrix, $[A/b]=\begin{bmatrix}1&1&1&6\\1&-1&2&5\\3&1&1&8\\2&-2&3&7\end{bmatrix}$

$R_2\to R_2-R_1$
$R_3\to R_3-3R_1$
$R_4\to R_4-2R_1$

$$[A/b]=\begin{bmatrix}1&1&1&6\\0&-2&1&-1\\0&-2&-2&-10\\0&-4&+1&-5\end{bmatrix}$$

$R_3\to R_3-R_2$
$R_4\to R_4-2R_2$

$$[A/b]=\begin{bmatrix}1&1&1&6\\0&-2&1&-1\\0&0&-3&-9\\0&0&-1&-3\end{bmatrix}$$

$R_4\to 3R_4-R_3$

$$[A/b]=\begin{bmatrix}1&1&1&6\\0&-2&1&-1\\0&0&-3&-9\\0&0&0&0\end{bmatrix}$$

$\rho(A/b)=3$ & $\rho(A)=3$

$\rho(A/b)=\rho(A)=3$

Given system is consistent & no. of unknowns $=3$.
Given system has a unique solution.

$$\begin{bmatrix}1&1&1\\0&-2&1\\0&0&-3\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-1\\-9\\0\end{bmatrix}$$

$x+y+z=6$
$-2y+z=-1$
$-3z=-9 \Rightarrow \boxed{z=3}$

$-2y+3=-1 \Rightarrow \boxed{y=2}$

$x+2+3=6 \Rightarrow \boxed{x=1}$

$\therefore x=1, y=2, z=3.$

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2. Find values of λ so that eqn's $x+y+z=1$, $2x+y+4z=\lambda$, $4x+y+10z=\lambda^2$ have a solution & solve them completely in each case.

$$AX=b \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

Augmented matrix, $[A|b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & \lambda \\ 4 & 1 & 10 & \lambda^2 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$ $\quad [A|b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & \lambda-2 \\ 0 & -3 & 6 & \lambda^2-4 \end{bmatrix}$

$R_3 \to R_3 - 3R_2$ $\quad [A|b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & \lambda-2 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{bmatrix}$

Given system has a solution if $\rho(A|b) = \rho(A)$

i.e., if $\lambda^2-3\lambda+2=0$

$\lambda^2-2\lambda-\lambda+2=0$

$\lambda(\lambda-2)-(\lambda-2)=0$

$\lambda=1,2$

<u>Case i</u> – If $\lambda=1$ then, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\left.\begin{aligned} x+y+z&=1 \\ -y+2z&=-1 \end{aligned}\right\}$ 2 sol's in 3 unknowns

Let $z=\alpha$

$-y+2\alpha=-1 \Rightarrow y=2\alpha+1$

$x+2\alpha+1+\alpha=1 \Rightarrow x=-3\alpha$

$\therefore x=-3\alpha,\ y=2\alpha+1,\ z=\alpha$ where α is arbitrary

Case 2: If $\lambda = 2$ then, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$x+y+z=1$

$-y+2z=0 \Rightarrow y=2z$

Let $z=\alpha$, $y=2\alpha$

$x+2\alpha+\alpha=1 \Rightarrow x=-3\alpha+1$

$\therefore x=-3\alpha+1,\ y=2\alpha,\ z=\alpha$ where α is arbitrary.

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3. Determine values of k for which the system of eqns $x-ky+z=0,\ kx+3y-kz=0,\ 3x+y-z=0$ has
i) Only zero solution ii) Non-zero solution.

Matrix form is $\begin{bmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i) Given system has only zero solution if $\rho(A)=3$

$|A| \neq 0$

$[-3+k]+k[-k+3k]+[k-9] \neq 0$

$k^2+k-6 \neq 0$

$(k+3)(k-2) \neq 0$

$k \neq 2$ & $k \neq -3$

ii) Given system has non-zero solution if $k=2 / k=-3$.

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4. Find values of a & b such that the equations $x+y+z=6,\ x+2y+3z=10,\ x+2y+az=b$ have
i) no solution ii) Unique solution iii) infinite solutn.

Matrix form $\Rightarrow AX=b \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ b \end{bmatrix}$

Augmented matrix: $[A/b]$ is $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$[A/b] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\quad [A/b] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{bmatrix}$

i) Given system has a unique solution if

$\rho(A/b) = \rho(A) = 3.$

i.e, if $a-3 \neq 0$ & b is arbitary

$\therefore a \neq 3$ & b is arbitary.

ii) Given system has no solution if

$\rho(A/b) \neq \rho(A)$

i.e, $a-3=0$ & $b-10 \neq 0$

$\therefore a=3$ & $b \neq 10$.

iii) Given system has infinite solution if

$\rho(A/b) = \rho(A) < 3$

if $a-3=0$ & $b-10=0$

$\therefore a=3$ & $b=10$.

---

5. Find all Eigen values & corresponding eigen vectors of matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Char eqn is $\lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0.$

$tr(A) = 6$

$A_{11} + A_{22} + A_{33} = 6+3+2 = 11$

$|A| = 1(6-0)+0+0 = 6.$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$

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$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \vert 0 \end{array} \Rightarrow (\lambda-1)(\lambda^2-5\lambda+6)=0$$

$$(\lambda-1)(\lambda-2)(\lambda-3)=0$$

$\lambda = 1,2,3$ are eigen values.

i) To find Eigen vectors corresponding to $\lambda=1$, take $Ax=x$ where $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$(A-I)x=0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$x_2+x_3=0$; $2x_1+2x_3=0 \Rightarrow x_1+x_3=0$

Let $x_3=\alpha$

$x_2=x_1=-\alpha$

Eigen vector is $x=\alpha\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

ii) To find Eigen vectors corresponding to $\lambda=2$ take $Ax=2x$ where $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$(A-2I)x=0 \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1=0 \Rightarrow x_1=0$

$x_3=0$

Let $x_2=\alpha$ where α is arbitrary.

Eigen vector is $x=\alpha\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

iii) To find Eigen vector corresponding to $\lambda=3$, take $Ax=3x$ where $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$(A-3I)x=0 \Rightarrow \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1=0$, $x_2=x_3$

Let $x_3=\alpha$ then $x_1=0$, $x_2=x_3=\alpha$

Eigen vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\alpha\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

6. Find eigen values & eigen vectors of $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Given: $A=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Char eqn is $\lambda^2 - tr(A)\lambda + |A| = 0$

$tr(A) = 1+1 = 2$

$|A| = 1+1 = 2$

$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$

$\lambda = 1+i, 1-i$

i) To find Eigen vector corresponding to $\lambda = 1+i$,

Take $(A-(1+i)I)x = 0 \Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-ix_1 + x_2 = 0 \Rightarrow x_1 + ix_2 = 0$ (Multiplied by i).

$x_1 + ix_2 = 0$

Let $x_2 = \alpha$, $x_1 = -i\alpha$

Eigen vector is $X = \alpha\begin{bmatrix} -i \\ 1 \end{bmatrix}$

ii) To find Eigen vector corresponding to $\lambda = 1-i$

Take $(A-(1-i)I)x = 0 \Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$ix_1 + x_2 = 0$ & $-x_1 + ix_2 = 0 \Rightarrow ix_1 + x_2 = 0$ (Multiplied with i)

Let $x_1 = \alpha$, $x_2 = -i\alpha$

Eigen Vector is $X = \alpha\begin{bmatrix} 1 \\ -i \end{bmatrix}$

---

7. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ then find A^{-1} by using Cayley-Hamilton theorem.

Given: $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

Char eqn is $\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$

$tr(A) = 2$, $A_{11}+A_{22}+A_{33} = -1-2+2 = -1$

$|A| = 2(-1) - 3(0) + 4(0) = -2$

$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

From CHT, $A^3 - 2A^2 - A + 2 = 0$.

$A^2 - 2A - I + 2A^{-1} = 0$

$A^{-1} = [-A^2 + 2A + I] \cdot \frac{1}{2}$

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$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4 & 6 & 8 \\ 0 & 2 & 10 \\ 0 & 0 & -2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 9 & 19 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4 & 6 & 8 \\ 0 & 2 & 10 \\ 0 & 0 & -2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 9 & 15 & 27 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

---

8. Find the characteristic equation of $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ & hence find the matrix $A^8 - 5A^7 - A^6 - 5A^5 - A^4 6A^2 + I$.

Given: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Char equ is $\lambda^2 - tr(A)\lambda + |A| = 0$

$tr(A) = 5$, $|A| = 4 - 6 = -2$.

$\lambda^2 - 5\lambda - 2 = 0$.

From CHT, $A^2 - 5A - 2I = 0$

$A^2 = 5A + 2I$.

$A^3 = 5A^2 + 2A = 5(5A+2I) + 2A = 25A + 10I + 2A = 27A + 10I$.

$A^4 = 5A^3 + 2A^2 = 5(27A + 10I) + 2(5A + 2I) = 135A + 10A + 54I$

$= 145A + 54I$

$A^5 = 145(A^2) + 54A = 145(5A + 2I) + 54A = 725A + 54A + 290I$

$= 779A + 290I$.

$A^6 = 779(5A + 2I) + 290A = 4185A + 1558I$.

$A^7 = 4185(5A + 2I) + 1558A = 22483A + 8370I$.

$A^8 = 22483(5A + 2I) + 8370A = 120785A + 44966I$.

$A^8 - 5A^7 - A^6 - 5A^5 - A^4 + 6A^2 + I = 120785A + 44966I - 112415A$

$- 41850I - 4185A - 1558I - 3895A - 1450I - 145A - 54I$

$+ 30A + 12I + I$

$= 175A + 67I$

$\therefore A^8 - 5A^7 - A^6 - 5A^5 - A^4 + 6A^2 + I = \begin{bmatrix} 175 & 525 \\ 350 & 700 \end{bmatrix} + \begin{bmatrix} 67 & 0 \\ 0 & 67 \end{bmatrix}$

$= \begin{bmatrix} 242 & 525 \\ 350 & 767 \end{bmatrix}$

9. If $A=\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Find eigen values of $3A^5-A^4+A^2+3I-A^{-1}$.

Char eqn is $\lambda^3-tr(A)\lambda^2+(A_{11}+A_{22}+A_{33})\lambda-|A|=0$

$tr(A)=2, \quad A_{11}+A_{22}+A_{33}=-3-1+3=-1$

$|A|=-2$

$\lambda^3-2\lambda^2-\lambda+2=0$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & 0 \end{array} \Rightarrow (\lambda-1)(\lambda^2-\lambda-2)=0$$

$(\lambda-1)(\lambda^2-2\lambda+\lambda-2)=0$

$(\lambda-1)[(\lambda-2)\lambda+\lambda-2]=0$

$(\lambda-1)((\lambda-2)(\lambda+1))=0$

$\lambda=-1,1,2.$

$\lambda=-1,1,2$

We know that if λ is eigen value of A then eigen value of $3A^5-A^4+A^2+3I-A^{-1}$ is $3\lambda^5-\lambda^4+\lambda^2+3-\lambda^{-1}$

If $\lambda=1$ then eigen value is 5

If $\lambda=-1$ then eigen value is 1

If $\lambda=2$ then eigen value is $\frac{173}{2}$

$\therefore$ Eigen values are $1, 5, \frac{173}{2}$.

10. Reduce the quadratic form $Q=2(xy+yz+zx)$ to Canonical form by orthogonal transformation & find its nature.

Given: $Q=2(xy+yz+zx)$ —①

Compare (1) with, $ax^2+by^2+cz^2+2hxy+2fyz+2gxz$

$2h=2, \quad 2f=2, \quad 2g=2$

$h=f=g=1.$

Symmetric matrix is $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Char eqn is $\lambda^3-tr(A)\lambda^2+(A_{11}+A_{22}+A_{33})\lambda-|A|=0.$

$tr(A) = 0$, $A_{11} + A_{22} + A_{33} = -1-1-1 = -3$, $|A| = -(-1)+1 = 2$

$\Rightarrow \lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$

$\lambda^3 - 3\lambda - 2 = 0$

$\lambda = -1, 2, -1$

i) To find X_1, for $\lambda = -1$ take $AX_1 = -X_1$

$(A+I)X_1 = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left.\begin{array}{l} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array}\right\}$ 1 equation in 3 knowns

Let $x_2 = \alpha$, $x_3 = \beta$ then $x_1 = -\alpha - \beta$

Let $\alpha = 2$, $\beta = 0$ then $X_1 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ & $\|X_1\| = \sqrt{4+4} = 2\sqrt{2}$.

Let $\alpha = 0$, $\beta = 2$ then $X_2 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ & $\|X_2\| = 2\sqrt{2}$.

$\dfrac{X_1}{\|X_1\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ & $\dfrac{X_2}{\|X_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

ii) To find X_3 for $\lambda = 2$, take $AX_3 = 2X_3 \Rightarrow (A - 2I)X_3 = 0$

$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 + x_2 + x_3 = 0$
$x_1 - 2x_2 + x_3 = 0$
$x_1 + x_2 - 2x_3 = 0$

$\dfrac{x_1}{3} = \dfrac{-x_2}{-3} = \dfrac{x_3}{3} \Rightarrow X_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ & $\|X_3\| = \sqrt{9+9+9} = 3\sqrt{3}$

$\dfrac{X_3}{\|X_3\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Model Matrix $B = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

$B^TAB = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Now, we use orthogonal transformation $X = BY$

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ then, $X^TAX = Y^T(B^TAB)Y = [y_1\ y_2\ y_3]\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$X^TAX = -y_1^2 - y_2^2 + 2y_3^2$.

$\therefore$ The given quadratic form is indefinite as it has +ve as well as -ve Eigen values.

11. Reduce the QF $2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3-2x_3x_1$ to canonical form through orthogonal transformation & find its rank, index & signature.

Given: $2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3-2x_3x_1$.

Compare (1) with $ax_1^2+bx_2^2+cx_3^2+2hx_1x_2+2fx_2x_3+2gx_3x_1$

$a=2,\ b=2,\ c=2,\ h=-1,\ f=-1,\ g=-1$

Symmetric matrix is $A=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Char eqn is $\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$

$tr(A)=6,\ A_{11}+A_{22}+A_{33} = 4-1+4-1+4-1=9$

$|A| = 2(4-1)+(-2-1)-(1+2) = 6-3-3=0$

$\Rightarrow \lambda^3-6\lambda^2+9\lambda=0 \Rightarrow \lambda(\lambda^2-6\lambda+9)=0$

$\lambda=0$ & $\lambda^2-6\lambda+9=0$

$\lambda^2-3\lambda-3\lambda+9=0$

$\lambda(\lambda-3)-3(\lambda-3)=0$

$(\lambda-3)(\lambda-3)=0$

Eigen values are: $\lambda=0,3,3$

i) To find X_1, for $\lambda=0 \Rightarrow AX_1=0$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1-x_2-x_3=0 \Rightarrow 2x_1-x_2-x_3=0$

$-x_1+2x_2-x_3=0 \Rightarrow x_1-2x_2+x_3=0$

$-x_1-x_2+2x_3=0 \Rightarrow x_1+x_2-2x_3=0$

$\dfrac{x_1}{4-1}=\dfrac{-x_2}{-2-1}=\dfrac{x_3}{1+2} \Rightarrow \dfrac{x_1}{3}=\dfrac{x_2}{3}=\dfrac{x_3}{3}$

$X_1=\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

$\|X_1\| = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$

$\dfrac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

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ii) To find X_2, X_3 for $\lambda=3 \Rightarrow AX_2=3X_2$

$(A-3I)X_2=0.$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1+x_2+x_3&=0 \\ x_1+x_2+x_3&=0 \\ x_1+x_2+x_3&=0 \end{aligned}\right\} \text{1 equ in 3 unknowns.}$$

Let $x_2=\alpha$, $x_3=\beta$, $x_1=-\alpha-\beta$

Let $\alpha=2$ & $\beta=4$, $X_2=\begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$

Let $\alpha=4$ & $\beta=2$, $X_3=\begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

$\|X_2\|=\|X_3\|=\sqrt{36+4+16}=\sqrt{56}=2\sqrt{14}$

$$\frac{X_2}{\|X_2\|}=\begin{bmatrix} -3/\sqrt{14} \\ 1/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix} \quad \& \quad \frac{X_3}{\|X_3\|}=\begin{bmatrix} -3/\sqrt{14} \\ 2/\sqrt{14} \\ 1/\sqrt{14} \end{bmatrix}$$

Now the model/orthogonal matrix is -

$$B=\begin{bmatrix} 1/\sqrt{3} & -3/\sqrt{14} & -3/\sqrt{14} \\ 1/\sqrt{3} & 1/\sqrt{14} & 2/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} & 1/\sqrt{14} \end{bmatrix}$$

$$B^TAB=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -3/\sqrt{14} & 1/\sqrt{14} & 2/\sqrt{14} \\ -3/\sqrt{14} & 2/\sqrt{14} & 1/\sqrt{14} \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & -3/\sqrt{14} & -3/\sqrt{14} \\ 1/\sqrt{3} & 1/\sqrt{14} & 2/\sqrt{14} \\ 1/\sqrt{3} & 2/\sqrt{14} & 1/\sqrt{14} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now we use orthogonal transformation $X=BY$

where $Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$X^TAX=Y^T(B^TAB)Y=[y_1\ y_2\ y_3]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$X^TAX=0y_1^2+3y_2^2+3y_3^2$$

Rank $=2$, Index $=2$, Signature $=2$.

Unit - II : Differential Equations of first order

SAQs :-

D-19-B (1) Under what conditions the equation $[f(x)+g(y)]dx+[h(x)+d(y)]dy=0$ is exact.

J-15-R (2) Find the values of 'a' and 'b' such that the equation $(3ax^2+2e^y)dx+(2bxe^y+3y)dy=0$ is exact.

D-16-B (3) Solve $x\,dy-y\,dx+y^2dx=0$.

M-19-R (4) Solve $y\,dx-x\,dy+e^{1/x}dx=0$.

D-17-R (5) Solve $(x^2+y^2+x)dx+xy\,dy=0$.

D-17-B (6) Solve $y(1+xy)dx+x(1-xy)dy=0$

J-17-R (7) Solve $(x^3-2y^2)dx+2xy\,dy=0$

J-17-B (8) Find an integrating factor of $(x^2y-2xy^2)dx+(3x^2y-x^3)dy=0$

J-16-B (9) Solve $(y\sin 2x)dx-(1+y^2+\cos^2 x)dy=0$

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J-17-R (10) Solve $\cos^2 x\dfrac{dy}{dx}+y=\tan x$.

D-16-B (11) Solve $\dfrac{dy}{dx}+xy=2x$

A-16-R (12) Solve $\dfrac{dy}{dx}-y\tan x=e^x\sec x$.

D-17-R (13) Solve $x\dfrac{dy}{dx}+2y-x^2\log x=0$

M-19-R (14) Find the orthogonal trajectories of $(x-c)^2+y^2=1$

D-19-B (15) Write Riccati's and Clairauts equations.

LAQs:-

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D-17-B
(1) Solve $y(2xy+e^x)\,dx - e^x\,dy = 0$

A-16-R
(2) Solve $(x^2y-2xy^2)\,dx + (3x^2y-x^3)\,dy = 0$.

J-16-B
(3) Solve $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$.

J-17-B
(4) Solve $2xy\,dy - (x^2+y^2+1)\,dx = 0$.

D-17, J-17-B
(5) Solve $\frac{dx}{dy} + x\sin 2y = x^3\cos^2 y$.

J-17-R
(6) Solve $x\frac{dy}{dx} + y = y^2x^3\cos x$.

M-19-R
(7) Solve the diff eqn $y' + 4xy + xy^3 = 0$.

D-17-R
(8) Find the general solution of $y' = 3y^2 - (1+6x)y + 3x^2 + x + 1$ if $y = x$ is a particular solution.

D-17-B, J-16-B
(9) Find the general solution of $y' = 2xy^2 + (1-4x)y + 2x - 1$ if $y = 1$ is a solution of it.

A-16-R
(10) Find the general solution of $y' = y^2 - (2x-1)y + (x^2-x+1)$ if $y = x$ is a soln of it.

D-15-R
(11) Obtain the general and singular solution of Clairauts equation $y = xy' - \frac{(y')^2}{2}$

D-19-B, J-17-B, J-16-B.
(12) Find the orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$ where λ is the parameter.

D-17-R
(13) Find the orthogonal trajectories of $x^{2/3} + y^{2/3} = a^{2/3}$ where 'a' is the parameter.

D-16-B
(14) Show that the family of curves $y^2 = 4a(a+x)$, a being parameter, is self orthogonal.

D-19-B
(15) Find the orthogonal trajectories of $r = a(1-\cos\theta)$.

* * *

Solutions

SAQs:

1. Under what conditions the equation $[f(x)+g(y)]dx + [h(x)+d(y)]dy = 0$ is exact.

Given: $[f(x)+g(y)]dx + [h(x)+d(y)]dy = 0. \quad -(1)$

This is in the form of $Mdx+Ndy=0$.

$M = f(x)+g(y), \quad N = h(x)+d(y)$

Eqn (1) is exact, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial g(y)}{\partial y} = \frac{\partial h(x)}{\partial x} + \frac{\partial d(y)}{\partial x}$$

$\therefore g'(y) = h'(x) + d'(y).\frac{dy}{dx}$ is required condition.

2. Find the values of 'a' & 'b' such that the equation $(3ax^2+2e^y)dx + (2bxe^y+3y)dy = 0$ is exact.

Given eqn is $(3ax^2+2e^y)dx + (2bxe^y+3y)dy = 0. \quad -(1)$

Given that eqn (1) is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$2e^y = 2be^y$$

$\therefore b = 1$ and a is any arbitrary real no.

3. Solve $xdy - ydx + y^2dx = 0$.

Given: $xdy - ydx + y^2dx = 0 \quad -(1)$.

This is in the form of $Mdx+Ndy=0$

where $M = y^2 - y, \quad N = x$.

$$\frac{\partial M}{\partial y} = 2y-1 \neq \frac{\partial N}{\partial x} = 1$$

Eqn (1) is non-exact.

Divide by x^2, $\frac{(xdy-ydx)}{x^2} + \frac{y^2}{x^2}dx = 0$.

$$d\left(\frac{y}{x}\right) + \frac{y^2}{x^2}dx = 0.$$

Integrating, $\int d\left(\frac{y}{x}\right) + \int \frac{y^2}{x^2}dx = c.$

$$\frac{y}{x} - \frac{y^2}{x} = c.$$

---

4. Solve $ydx - xdy + e^{1/x}dx = 0$.

Given: $ydx - xdy + e^{1/x}dx = 0$ —①.

$(e^{1/x}+y)dx - xdy = 0 \Rightarrow Mdx + Ndy = 0.$

$M = e^{1/x}+y, \quad N = -x$

$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = -1$

Equ (1) is non-exact.

Divide by x^2, $\left(\frac{xdy-ydx}{x^2}\right) - \frac{e^{1/x}}{x^2}dx = 0.$

$$d\left(\frac{y}{x}\right) - \frac{e^{1/x}}{x^2}dx = 0.$$

Integrating, $\int d\left(\frac{y}{x}\right) - \int \frac{e^{1/x}}{x^2}dx = c$

let $1/x = t \Rightarrow -\frac{1}{x^2}dx = dt.$

$$\frac{y}{x} + \int e^t dt = c$$

$$\therefore \frac{y}{x} + e^{1/x} = c$$

---

5. Solve $(x^2+y^2+x)dx + xydy = 0.$

Given: $(x^2+y^2+x)dx + xydy = 0$ —①

This is in the form of $Mdx + Ndy = 0.$

$\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = y.$

Equ ① is non-exact also non-homogeneous.

Now, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x} = f(x)$

$$IF = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log_e x} = x.$$

$$IF \times (1) \Rightarrow (x^3 + xy^2 + x^2)dx + x^2y\,dy = 0 \quad \text{— (2)}.$$

$$M_1 dx + N_1 dy = 0.$$

$$\frac{\partial M_1}{\partial y} = 2xy \ , \quad \frac{\partial N_1}{\partial x} = 2xy$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eqn (2) is exact.

$$G.S: \underset{y\text{-constant}}{\int} M\,dx + \int \left(\begin{array}{l}\text{Terms of } N\\ \text{not involving } x\end{array}\right) dy = c.$$

$$\int x^3 + xy^2 + x^2\,dx + \int 0\,dy = c$$

$$\therefore \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c.$$

---

6. Solve $y(1+xy)\,dx + x(1-xy)\,dy = 0.$

Given: $y(1+xy)dx + x(1-xy)dy = 0$ — (1)

$$\frac{\partial M}{\partial y} = 1 + 2xy \neq \frac{\partial N}{\partial x} = 1 - 2xy$$

Eqn (1) is non-exact & also non-homogeneous

It is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0.$

$$\therefore IF = \frac{1}{Mx - Ny} = \frac{1}{(y + xy^2)x - (x - x^2y)y}$$

$$= \frac{1}{\cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2} = \frac{1}{2x^2y^2}$$

$$IF \times (1) \Rightarrow \left(\frac{y + xy^2}{2x^2y^2}\right)dx + \left(\frac{x - x^2y}{2x^2y^2}\right)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \quad \text{— (2)}$$

It is in $M_1 dx + N_1 dy = 0$ form.

$$\frac{\partial M_1}{\partial y} = -\frac{1}{2x^2y^2} = \frac{\partial N_1}{\partial x} = -\frac{1}{2x^2y^2}$$

Eqn (2) is exact.

G.S: $\int\limits_{y\text{-cnst}} M\,dx + \int \left(\begin{array}{c}\text{terms of } N \\ \text{not involving } x\end{array}\right) dy = c.$

$$\int \left(\frac{1}{2x^2 y} + \frac{1}{2x}\right) dx - \int \frac{1}{2y}\, dy = c.$$

$$\frac{1}{2y}\frac{x^{-2+1}}{(-2+1)} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\frac{-1}{2xy} + \frac{1}{2}\log\left(\frac{x}{y}\right) = c.$$

---

7. Solve $(x^3 - 2y^2)dx + 2xy\,dy = 0.$

Given: $(x^3 - 2y^2)dx + 2xy\,dy = 0. \quad -(1)$

This is of the form $M dx + N dy = 0.$

$$\frac{\partial M}{\partial y} = -4y, \quad \frac{\partial N}{\partial x} = 2y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Equ (1) is non-exact and also non-homogen

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y - 2y}{2xy} = \frac{-6y}{2xy} = -\frac{3}{x} = f(x).$$

$$\therefore IF = e^{\int f(x)dx} = e^{-\int \frac{3}{x}dx} = e^{-3\log x} = e^{\log x^{-3}} = \frac{1}{x^3}$$

$$IF \times (1) \Rightarrow \left(1 - \frac{2y^2}{x^3}\right)dx + \frac{2y}{x^2}dy = 0 \quad -(2)$$

This is in the form of $M_1 dx + N_1 dy = 0.$

$$\frac{\partial M_1}{\partial y} = -\frac{4y}{x^3} = \frac{\partial N_1}{\partial x} = \frac{-4y}{x^3}$$

Equ (2) is exact.

G.S: $\int\limits_{y\text{-cnst}} M_1 dx + \int \left(\begin{array}{c}N_1 \text{ terms} \\ \text{not involving } x\end{array}\right) dy = c$

$$\int \left(1 - \frac{2y^2}{x^3}\right)dx + \int 0\,dy = c.$$

$$x - 2y^2\left(\frac{x^{-3+1}}{-3+1}\right) = c$$

$$\therefore x + 4\frac{y^2}{x^2} = c.$$

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8. Find an integrating factor of $(x^2y-2xy^2)dx + (3x^2y-x^3)dy=0$.

Given: $(x^2y-2xy^2)dx+(3x^2y-x^3)dy=0$ —— ①.

This is in the form of $Mdx+Ndy=0$.

$$\frac{\partial M}{\partial y} = x^2-4xy, \quad \frac{\partial N}{\partial x} = 6xy-3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eqn ① is non-exact. and it is homogeneous differential eqn.

$$IF = \frac{1}{Mx+Ny} = \frac{1}{(x^2y-2xy^2)x+(3x^2y-x^3)y}$$

$$= \frac{1}{x^3y-2x^2y^2+3x^2y^2-x^3y}$$

$$= \frac{1}{x^2y^2}$$

$\therefore$ Integrating factor $= \frac{1}{x^2y^2}$.

---

9. Solve $(y\sin 2x)dx-(1+y^2+\cos^2x)dy=0$.

Given: $(y\sin 2x)dx-(1+y^2+\cos^2x)dy=0$ —— ①

This is in the form of $Mdx+Ndy=0$.

where $M=y\sin 2x$ and $N=-(1+y^2+\cos^2x)$

$$\frac{\partial M}{\partial y} = \sin 2x, \quad \frac{\partial N}{\partial x} = -2\cos x(-\sin x) = \sin 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn ① is exact.

G.S: $\int\limits_{y\,const} y\sin 2x\,dx + \int(-1-y^2)\,dy = c$.

$$-y\frac{\cos 2x}{2} - y - \frac{y^3}{3} = c.$$

$$\frac{y}{2}\cos 2x + y + \frac{y^3}{3} = k \quad \text{where } k=-c.$$

10. Solve $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.

Given: $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \sec^2 x \cdot y = \frac{\tan x}{\cos^2 x}$$

This is a linear equation of the form

$\dfrac{dy}{dx} + Py = Q$ where $P = \sec^2 x$ & $Q = \dfrac{\tan x}{\cos^2 x}$.

$\therefore IF = e^{\int P dx} = e^{\tan x}$.

G.S: $y \times IF = \int (IF \times Q) dx + c$

$$y e^{\tan x} = \int e^{\tan x} \cdot \frac{\tan x}{\cos^2 x} dx + c$$

Let $\tan x = t$

$\sec^2 x \, dx = dt$.

$$y e^{\tan x} = \int e^t \cdot t \, dt + c$$

$$y e^{\tan x} = e^t (t-1) + c = e^{\tan x}(\tan x - 1) + c$$

$$\therefore e^{\tan x} y = (\tan x - 1) e^{\tan x} + c$$

---

11. Solve $\dfrac{dy}{dx} + xy = 2x$.

This is of the form $\dfrac{dy}{dx} + Py = Q$ where $P = x$, $Q = 2x$.

$\therefore IF = e^{\int P dx} = e^{\int x dx} = e^{x^2/2}$.

G.S: $IF \times y = \int (IF \cdot Q) dx + c$

$$y e^{\frac{x^2}{2}} = 2\int e^{\frac{x^2}{2}} \cdot x \, dx + c$$

let $\dfrac{x^2}{2} = t \Rightarrow \dfrac{2x}{2} dx = dt \Rightarrow 2x dx = 2dt$

$$y e^{\frac{x^2}{2}} = 2\int e^t dt + c = 2e^t + c$$

$$\therefore y e^{x^2/2} = 2e^{x^2/2} + c.$$

12. Solve $\frac{dy}{dx} - y\tan x = e^x \sec x$.

This is in the form of $\frac{dy}{dx} + Py = Q$

where $P = -\tan x$, $Q = e^x \sec x$.

$\therefore IF = e^{\int P dx} = e^{-\int \tan x dx} = e^{-\log|\sec x|} = \cos x$

$\therefore$ G.S: $y \times IF = \int (IF \times Q) dx + c$

$y\cos x = \int e^x \sec x \cos x dx + c$

$\therefore y\cos x = e^x + c$.

---

13. Solve $x\frac{dy}{dx} + 2y - x^2\log x = 0$.

$\frac{dy}{dx} + \frac{2y}{x} - x\log x = 0$.

$\frac{dy}{dx} + \frac{2}{x} \cdot y = x\log x$

This is linear differential Equation of the form $\frac{dy}{dx} + Py = Q$ where $P = \frac{2}{x}$, $Q = x\log x$

$\therefore IF = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$

G.S: $IF \times y = \int (IF \times Q) dx + c$

$x^2 y = \int x^3 \log x dx + c$

$= \log x \cdot \frac{x^4}{4} - \frac{x^4}{16} + c$.

$\therefore x^2 y = \frac{x^4}{4}\left(\log x - \frac{1}{4}\right) + c$.

---

15. Write Riccati's & Clairaut's equations.

RICCATI'S EQUATION - An equation of the form $y' = P(x)y^2 + q(x)y + r(x)$ is called Riccati's equation

CLAIRAUT'S EQUATION - An equation of the form $y = y'x + f(y')$ (or) $y = px + f(p)$ where $y' = p$ is called clairaut's eqn.

LAQ's :

1. Solve $y(2xy+e^x)dx - e^x dy = 0$.

Given: $y(2xy+e^x)\,dx - e^x\,dy = 0$. —①

$$e^x dy = (2xy^2 + e^x y)\,dx$$

$$\frac{dy}{dx} = \frac{2x}{e^x}\cdot y^2 + y.$$

$$\frac{dy}{dx} - y = \frac{2x}{e^x} y^2$$ which is Bernoulli's eq^n with $n=2$.

Divide by y^2, $y^{-2}\frac{dy}{dx} - y^{-1} = \frac{2x}{e^x}$ —② .

Put $y^{-1} = z \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

From (2), $-\frac{dz}{dx} - z = \frac{2x}{e^x} \Rightarrow \frac{dz}{dx} + z = -\frac{2x}{e^x}$ —③

which is linear eq^n in z.

$\therefore IF = e^{\int dx} = e^x$

G.S of (3): $ze^x = \int \cancel{e^x}\left(-\frac{2x}{\cancel{e^x}}\right)dx + c$

$$ze^x = -\frac{2x^2}{2} + c.$$

G.S of (1) : $\frac{e^x}{y} = -x^2 + C.$

---

2. Solve $(x^2y - 2xy^2)dx + (3x^2y - x^3)\,dy = 0$.

Given: $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$ —①

This is in the form of $Mdx + Ndy = 0$.

$\frac{\partial M}{\partial y} = x^2 - 4xy \neq \frac{\partial N}{\partial x} = 6xy - 3x^2$

$\therefore$ Eq^n ① is non-exact

Eq^n ① is homogeneous differential equation

$$IF = \frac{1}{Mx+Ny} = \frac{1}{(x^2y-2xy^2)x + (3x^2y - x^3)y} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

$IF \times (1) \Rightarrow \left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$ — ②

$M_1 dx + N_1 dy = 0$.

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2} = \frac{\partial N_1}{\partial x} = -\frac{1}{y^2}$$

$\therefore$ Eqn (2) is exact

G.S: $\int M_1 dx + \int \left(\begin{matrix} N_1 \text{ terms not} \\ \text{involving } x \end{matrix}\right) dy = c$

first

$$\int \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y} dy = c$$

$$\therefore \frac{x}{y} - 2\log x + 3\log y = c.$$

---

3. Solve $(xy^3+y)dx + 2(x^2y^2+x+y^4)\,dy = 0$ — ①.

This is in the form of $Mdx+Ndy=0$.

$$\frac{\partial M}{\partial y} = 3xy^2+1 \quad \& \quad \frac{\partial N}{\partial x} = 2(2xy^2+1) = 4xy^2+2$$

Eqn (1) is non exact as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

and also, non-homogeneous.

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3xy^2+1-4xy^2-2}{2(x^2y^2+x+y^4)} = \frac{-(xy^2+1)}{2(x^2y^2+x+y^4)}$$

$\neq f(x)$ (or) k.

$$\Rightarrow \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4xy^2+2-3xy^2-1}{xy^3+y} = \frac{xy^2+1}{y(xy^2+1)} = \frac{1}{y} = f(y)$$

$$\therefore IF = e^{\int f(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y.$$

$IF \times (1) = (xy^4+y^2)dx + 2(x^2y^3+xy+y^5)\,dy = 0$ — ②

$M_1 dx + N_1 dy = 0$.

$$\frac{\partial M_1}{\partial y} = 4xy^3+2y = \frac{\partial N_1}{x} = 4xy^3+2y.$$

Eqn (2) is exact

G.S: $\int xy^4+y^2dx + 2\int y^5 dy = c$

$$\therefore \frac{x^2}{2}y + y^2x + \frac{y^6}{3} = c$$

4. Solve $2xy\,dy-(x^2+y^2+1)dx=0$.

Given: $2xy\,dy-(x^2+y^2+1)\,dx=0$ — ①

This is in the form of $M\,dx+N\,dy=0$.

$$\frac{\partial M}{\partial y}=-2y \neq \frac{\partial N}{\partial x}=2y.$$

Eqn ① is non-exact & non-homogeneous.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-2y-2y}{2xy}=\frac{-4y}{2xy}=-\frac{2}{x}=f(x)$$

$$IF=e^{\int f(x)dx}=e^{-2\int\frac{1}{x}dx}=\frac{1}{x^2}$$

$$IF\times(1)\Rightarrow \frac{2y}{x}dy-\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx=0 \quad \text{— ②}$$

$$\frac{\partial M_1}{\partial y}=-\frac{2y}{x^2}=\frac{\partial N_1}{\partial x}=-\frac{2y}{x^2}$$

Eqn (2) is exact

G.S: $\int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx+\int 0\,dy=c$

$$\therefore x-\frac{y^2}{x}-\frac{1}{x}=c.$$

---

5. Solve $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$

Given: $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$.

$$\frac{dy}{dx}+2x\sin y\cos y=x^3\cos^2 y.$$

$$\sec^2 y\frac{dy}{dx}+2x\tan y=x^3 \quad (\text{Divide by } \cos^2 y)$$

Let $\tan y=t \Rightarrow \sec^2 y\frac{dy}{dx}=\frac{dt}{dx}$

$\frac{dt}{dx}+2xt=x^3$ which is Linear eqn in 't'

$$\therefore IF=e^{\int 2x\,dx}=e^{x^2}$$

G.S: $te^{x^2}=\int e^{x^2}x^3dx+c=\int e^{x^2}x^2\cdot x\,dx+c$

let $x^2=u \Rightarrow 2x\,dx=du$

$$\Rightarrow te^{x^2}=\frac{1}{2}\int e^u u\,du+c=\frac{1}{2}e^u(u-1)+c.$$

$$\tan y\, e^{x^2}=\frac{1}{2}e^{x^2}(x^2-1)+c.$$

6. Solve $x\frac{dy}{dx} + y = y^2x^3\cos x$.

Given: $x\frac{dy}{dx} + y = y^2x^3\cos x$ — ①

$$\frac{dy}{dx} + \frac{y}{x} = y^2x^2\cos x$$

This is Bernoulli's eqn with $n=2$

Divide by y^2, $y^{-2}\frac{dy}{dx} + y^{-1}\frac{1}{x} = x^2\cos x$ — ②

Let $y^{-1} = z \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$

From (2), $-\frac{dz}{dx} + \frac{z}{x} = x^2\cos x$

$$\frac{dz}{dx} - \frac{z}{x} = -x^2\cos x \quad \text{— ③}$$

This is linear eqn in 'z'

$\therefore$ IF $= e^{\int -1/x\, dx} = e^{-\log x} = 1/x$.

G.S of (3): $\frac{z}{x} = \int \frac{1}{x}(-x^2\cos x)dx + c$

$\frac{z}{x} = -\int x\cos x\, dx + c = -[x\sin x + \cos x] + c$

G.S of (1): $\frac{1}{xy} = -[x\sin x + \cos x] + c$.

7. Solve the differential eqn $y' + 4xy + xy^3 = 0$.

Given: $y' + 4xy + xy^3 = 0$.

$$\frac{dy}{dx} + 4xy = -xy^3 \quad \text{— ①}$$

This is Bernoulli's eqn with $n=3$

Divide by y^3, $y^{-3}\frac{dy}{dx} + 4xy^{-2} = -x$ — ②.

Let $y^{-2} = z \Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow y^{-3}\frac{dy}{dx} = -\frac{dz}{2dx}$

From (2), $-\frac{dz}{2dx} + 4xz = -x$

$\frac{dz}{dx} - 8xz = 2x$ — ③ which is L.E in 'z'

$\therefore$ IF $= e^{\int -8x\,dx} = e^{-8x^2/2} = e^{-4x^2}$

G.S of (3), $ze^{-4x^2} = \int e^{-4x^2} \cdot 2x\,dx + c$

Let $-4x^2 = t \Rightarrow -8x\,dx = dt \Rightarrow 2x\,dx = -\frac{dt}{4}$

$$ze^{-4x^2} = -\frac{\int e^t\,dt}{4} + c$$

$$ze^{-4x^2} = -\frac{e^{-4x^2}}{4} + c.$$

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G.S of (1), $\frac{e^{-4x^2}}{y^2} = \frac{-e^{-4x^2}}{4} + c.$

$$\therefore \frac{e^{-4x^2}}{y^2} = -\frac{e^{-4x^2}}{4} + c$$

---

8. Find general solution of $y' = 3y^2 - (1+6x)y + 3x^2 + x + 1$ if $y = x$ is a particular solution.

Given is a Riccati's eqn of the form $y' = py^2 + qy + r$ & given that $y = x$ is a sol^n of it.

Put $y = x + \frac{1}{z}$

$$y' = 1 - \frac{z'}{z^2}$$

From given eqn, $1 - \frac{z'}{z^2} = 3\left(x + \frac{1}{z}\right)^2 - (1+6x)\left(x + \frac{1}{z}\right) + 3x^2 + x + 1$

$$1 - \frac{z'}{z^2} = 3x^2 + \frac{3}{z^2} + \frac{6x}{z} - x - \frac{1}{z} - 6x^2 - \frac{6x}{z} + 3x^2 + x + 1$$

$$-\frac{z'}{z^2} = \frac{3}{z^2} - \frac{1}{z}$$

$\frac{dz}{dx} = -3 + z \Rightarrow \frac{dz}{dx} - z = -3$ —(1) which is linear in 'z'.

IF $= e^{-x}$

G.S of (1) is : $ze^{-x} = -3\int e^{-x}dx + c$

$$ze^{-x} = 3e^{-x} + c$$

$$z = 3 + ce^{x}$$

G.S of given eqn : $y = x + \frac{1}{3 + ce^{x}}$.

9. Find the general solution of $y' = 2xy^2 + (1-4x)y + 2x - 1$ if $y=1$ is a solution of it.

Given: $y' = 2xy^2 + (1-4x)y + 2x - 1$ — ① which is a Riccati's eq^n of the form $y' = py^2 + qy + r$; and given $y=1$ is a sol^n of ①.

Put $y = 1 + \frac{1}{z} \Rightarrow y' = -\frac{z'}{z^2}$

From ①, $-\frac{z'}{z^2} = 2x\left(1+\frac{1}{z}\right)^2 + (1-4x)\left(1+\frac{1}{z}\right) + 2x - 1$

$$-\frac{z'}{z^2} = 2x + \frac{2x}{z^2} + \frac{4x}{z} + 1 + \frac{1}{z} - 4x - \frac{4x}{z} + 2x - 1$$

$$-\frac{z'}{z^2} = \frac{2x}{z^2} + \frac{1}{z}$$

$$z' = -2x - z \Rightarrow \frac{dz}{dx} + z = -2x \quad \text{— ②}$$

which is linear eq^n in 'z'.

$\therefore IF = e^{\int dx} = e^x$

G.S of ② is $ze^x = \int e^x(-2x)\,dx + c$

$$ze^x = -2(xe^x - e^x) + c$$

$$z = 2(1-x) + ce^{-x}$$

$\therefore$ G.S of (1) is $y = 1 + \frac{1}{z} = 1 + \frac{1}{2(1-x) + ce^{-x}}$

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10. Find the G.S of $y' = y^2 - (2x-1)y + (x^2 - x + 1)$ if $y = x$ is a sol^n of it.

Given: $y' = y^2 - (2x-1)y + (x^2 - x + 1)$ — ① which is a Riccati's eq^n of form $y' = py^2 + qy + r$

Given that $y = x$ is a sol^n of ①.

Put $y = x + \frac{1}{z} \Rightarrow y' = 1 - \frac{z'}{z^2}$

From ①, $1 - \frac{z'}{z^2} = \left(x + \frac{1}{z}\right)^2 - (2x-1)\left(x + \frac{1}{z}\right) + x^2 + 1 - x$.

$$1 - \frac{z'}{z^2} = x^2 + \frac{1}{z^2} + \frac{2x}{z} - 2x^2 - \frac{2x}{z} + x + \frac{1}{z} + x^2 - x + 1.$$

$-\frac{z'}{z^2} = \frac{1}{z^2} + \frac{1}{z} \Rightarrow z' = -1 - z$

$\frac{dz}{dx} + z = -1$ —② which is linear eqn in 'z'.

$\therefore IF = e^{\int dx} = e^x$

G.S of (2): $ze^x = -\int e^x dx + c$

$ze^x = -e^x + c$

$z = -1 + ce^{-x}$

G.S of (1): $y = x + \frac{1}{-1+ce^{-x}}$.

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11. Obtain the general & singular solution of Clairauts eqn $y = xy' - \frac{(y')^2}{2}$.

Given: $y = xy' - \frac{(y')^2}{2}$ —①

Let $y' = p$

From ①, $y = px - \frac{p^2}{2}$ which is a Clairauts' eq

G.S of (1) is $y = cx - \frac{c^2}{2}$ —②

Differentiate (2) w.r.t 'c', $x - c = 0 \Rightarrow c = x$

From (2), $y = x^2 - \frac{x^2}{2}$ which is singular solⁿ.

$\therefore$ G.S : $y = cx - c^2/2$

singular solⁿ : $y = x^2 - x^2/2$.

---

12. Find the Orthogonal Trajectories of $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$

Given family of curve : $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$ —①

Diff w.r.t 'x', $\frac{2x}{a^2} + \frac{2y}{a^2+\lambda}\frac{dy}{dx} = 0$.

$\frac{x}{a^2} + \frac{y}{a^2+\lambda}\frac{dy}{dx} = 0$.

$\frac{y}{a^2+\lambda} = \frac{-x}{a^2\left(\frac{dy}{dx}\right)}$

From ①, $\frac{x^2}{a^2} - \frac{xy}{a^2\left(\frac{dy}{dx}\right)} = 1.$

$x^2 - \frac{xy}{\left(\frac{dy}{dx}\right)} = a^2$ — ② which is diff equ of ①.

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$, $x^2 + \frac{xy}{(dx/dy)} = a^2$

$xy\left(\frac{dy}{dx}\right) = a^2 - x^2$ — ③

which is diff equ of OTs.

$y\,dy = \frac{a^2 - x^2}{x}\,dx \Rightarrow y\,dy = \left(\frac{a^2}{x} - x\right)dx$

G.S: $\int y\,dy = \int\left(\frac{a^2}{x} - x\right)dx + c$

$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c$

$\therefore \frac{x^2}{2} + \frac{y^2}{2} = a^2 \log x + c.$

---

13. Find the OTs of $x^{2/3} + y^{2/3} = a^{2/3}$ where 'a' is parameter.

Given: $x^{2/3} + y^{2/3} = a^{2/3}$ — ①

Diff w.r.t 'x', $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$ — ② which is diff equ of ①.

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$, $x^{-1/3} - y^{-1/3}\frac{dx}{dy} = 0$ — ③

which is diff equ of OTs.

$x^{-1/3} = y^{-1/3}\frac{dx}{dy}.$

$y^{1/3}\,dy = x^{1/3}\,dx$

G.S: $\int y^{1/3}\,dy = \int x^{1/3}\,dx$

$\therefore x^{4/3} = y^{4/3} + c.$

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14. Show that the family of curves $y^2=4a(a+x)$, a being parameter, is self orthogonal.

Given: $y^2 = 4a(a+x)$ — ①

Diff w.r.t 'x', $2y\frac{dy}{dx} = 4a(1+0) \Rightarrow y\frac{dy}{dx} = 2a$

$$a = y/_2 \, dy/dx$$

From (1), $y^2 = 4\, y/_2 \, dy/dx \,(x + y/_2 \, dy/dx)$

$$y = 2\, dy/dx\,(x + y/_2 \, dy/dx)$$

$$y = 2x\,dy/dx + y\,(dy/dx)^2$$

$y(dy/dx)^2 + 2x\,dy/dx - y = 0$ — ② which is diff eqn of (1).

Replace dy/dx with $-dx/dy$

$$y(-dx/dy)^2 + 2x(-dx/dy) - y = 0.$$

$$y(dx/dy)^2 - 2x\,dx/dy - y = 0.$$

Divide with $(dx/dy)^2$, $y - 2x \cdot \frac{1}{(dx/dy)} - y \cdot \frac{1}{(dx/dy)^2} = 0.$

$$y - 2x\frac{dy}{dx} - y\left(\frac{dy}{dx}\right)^2 = 0.$$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ — ③}$$

which is diff eqn of OTs.

It is observed that (2) & (3) are same

∴ Given family is self orthogonal.

---

15. Find the Orthogonal Trajectories of $r = a(1-\cos\theta)$

Given: $r = a(1-\cos\theta)$ — ①

Diff w.r.t 'θ', $\frac{dr}{d\theta} = a\sin\theta$

$$a = \frac{dr}{d\theta \cdot \sin\theta}$$

$$r = \frac{dr}{d\theta}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \frac{dr}{d\theta}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right)$$

$r = \frac{dr}{d\theta}\, \tan\theta/2$ which is diff equ of ① — ②

Replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$

$r = -r^2\frac{d\theta}{dr}\tan\theta/2 \Rightarrow -r\frac{d\theta}{dr}\tan\theta/2 = 1.$ — ③

which is a diff equ of OTs.

G.S : $\int \tan\theta/2\, d\theta = -\int \frac{1}{r}\, dr$

$$\frac{\log|\sec\theta/2|}{1/2} = -\log r + \log c.$$

$$2\log|\sec\theta/2| = \log c - \log r = \log c/r$$

$$\therefore r\sec^2\theta/2 = c.$$

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Unit-III

Differential Equations of higher order

SAQs:-

M-19-R (1) Find the solution of the initial value problem $y''+4y'+13y=0$; $y(0)=0$; $y'(0)=1$

D-17-B (2) Find the solution of the initial value problem $4y''-8y'+3y=0$; $y(0)=1$; $y'(0)=3$.

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D-17-R (3) Solve $(D^4-5D^2-36)y=0$

D-17-B (4) Solve $(D^3-3D+2)y=0$ where $D=\frac{d}{dx}$

J-17-R (5) Solve $(D^3+16D)y=0$

J-17-B (6) Solve $(D^3+2D^2-8D)y=0$ where $D=\frac{d}{dx}$

J-16-R (7) Solve $(D^4-81)y=0$, where $D=\frac{d}{dx}$.

D-17-R (8) Find particular integral of $(D^2+4D+4)y=e^{-2x}$

D-17-B (9) Find particular integral of $(D^3+16D)y=\sin 4x$.

A-16-R (10) Find particular integral of $\frac{d^3y}{dx^3}-y=(e^x+1)^2$

J-16-R (11) Find particular integral of $(D^2-4)y=\cos^2 x$.

J-17-B (12) Find particular integral of $(D^2-6D+9)y=18+54x$.

J-15-R (13) Find particular integral of $(D^2-a^2)y=x^4$.

D-16-B (14) Determine whether the functions $x^2, \frac{1}{x^2}$ are linearly independent on $(0,\infty)$

D-19-B (15) If $\frac{1}{x}$ is a solution of $x^2y''+4xy'+2y=0$. Then find the second linearily independent solution and write the general solution.

LAQs:-

D-17-R (1) Solve $(D^2-4D+3)y = 3e^x \cos 2x$.

D-17-B (2) Solve $(D^2+2D+1)y = xe^{-x}$

J-17-R (3) Solve $(D^2+4)y = x^2+1+\cos 2x$.

D-19-R (4) Show that e^x, e^{2x}, e^{3x} are the fundamental solutions of $y'''-6y''+11y'-6y=0$ on any interval I.

M-19-R A-16-R (5) Find the general solution of $y''+16y = 32\sec 2x$ by method of variation of parameters.

D-19-R (6) Find the general solution of $y''+y = \sec x$ by method of variation of parameters.

D-17-B (7) Solve $(D^2+1)y = x\cos x$ by method of variation of parameters.

J-16-B (8) Solve $y''+4y = 4\tan 2x$ by method of variation of parameters

D-17-B (9) Solve $(x^2D^2-xD-3)y = x^2\log x$.

D-17-R (10) Solve $(x^2D^2+xD-1)y = x^3$

D-19-R (11) Find the general solution of $x^3y'''-3xy'-3y = 16x+9x^2\log x$

J-17-B (12) Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y = x^2$

J-15-R (13) If $y_1 = e^{2x}$ is a solution of $y''-5y'+6y=0$, find second linearly independent solution.

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Solutions

SAQs:

1. Find the solution of initial value problem $y''+4y'+13y=0$; $y(0)=0$; $y'(0)=1$.

Given: $y''+4y'+13y=0$
$D^2y+4Dy+13y=0 \Rightarrow (D^2+4D+13)y=0$.

AE: $m^2+4m+13=0$.

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{16-52}}{2}=\frac{-4\pm\sqrt{-36}}{2}=\frac{-4\pm i(6)}{2}$$

$\therefore m=-2\pm 3i$

CF: $y_c=e^{-2x}(c_1\cos 3x+c_2\sin 3x)$.

$$y'=c_1(e^{-2x}(-2)\cos 3x-\sin 3x(3)e^{-2x})+c_2(-2e^{-2x}\sin 3x+3e^{-2x}\cos 3x)$$
$$=-2c_1e^{-2x}\cos 3x-3c_1e^{-2x}\sin 3x-2c_2e^{-2x}\sin 3x+3c_2e^{-2x}\cos 3x$$
$$=e^{-2x}\cos 3x(3c_2-2c_1)-e^{-2x}\sin 3x(3c_1+2c_2)$$

Given: $y(0)=0 \Rightarrow e^0(c_1\cos 0+c_2\sin 0)=0 \Rightarrow c_1=0$.

$y'(0)=1 \Rightarrow e^0\cos 0(3c_2-2c_1)-0=1 \Rightarrow 3c_2=1+2c_1$
$3c_2=1+0$
$c_2=\frac{1}{3}$.

$\therefore y=e^{-2x}(0)\cos 3x+\frac{1}{3}e^{-2x}\sin 3x$.
$y=\frac{1}{3}e^{-2x}\sin 3x$.

2. Find the solution of initial value problem $4y''-8y'+3y=0$; $y(0)=1$; $y'(0)=3$.

Given: $4y''-8y'+3y=0$; $y(0)=1$; $y'(0)=3$
$4D^2y-8Dy+3y=0 \Rightarrow (4D^2-8D+3)y=0$.

AE: $4m^2-8m+3=0 \Rightarrow 4m^2-2m-6m+3=0$.
$2m(2m-1)-3(2m-1)=0$
$(2m-3)(2m-1)=0$
$m=\frac{3}{2},\frac{1}{2}$

CF: $y_c=c_1e^{x/2}+c_2e^{3x/2}$.
$y'=\frac{1}{2}c_1e^{x/2}+\frac{3}{2}c_2e^{3x/2}$.

Given: $y(0)=1 \Rightarrow c_1+c_2=1$ —— (1)
$y'(0)=3 \Rightarrow \frac{1}{2}c_1+\frac{3}{2}c_2=3 \Rightarrow c_1+3c_2=6$ —— (2)

solving (1) & (2), we get $c_1=-\frac{3}{2}$, $c_2=\frac{5}{2}$

$\therefore y = -\frac{3}{2}e^{x/2} + \frac{5}{2}e^{\frac{3x}{2}}$.

3. Solve $(D^4-5D^2-36)y=0$.

Given: $(D^4-5D^2-36)y=0$

AE: $m^4-5m^2-36=0 \Rightarrow m^4-9m^2+4m^2-36=0$

$\Rightarrow m^2(m^2-9)+4(m^2-9)=0$

$\Rightarrow (m^2+4)(m^2-9)=0$

$\Rightarrow m^2=-2^2, \quad m^2=9$

$\Rightarrow m=\pm 2i, \quad m=\pm 3$

GS: $y=e^{0x}(c_1\cos 2x+c_2\sin 2x)+c_3e^{-3x}+c_4e^{3x}$.

4. Solve $(D^3-3D+2)y=0$

Given: $(D^3-3D+2)y=0$

AE: $m^3-3m+2=0$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 1\,0 \end{array}$$

$\Rightarrow m^2+m-2=0$

$\Rightarrow m^2+2m-m-2=0$

$\Rightarrow m(m+2)-(m+2)=0$

$m=1,1,-2$

GS: $(c_1+c_2x)e^{x}+c_3e^{-2x}$

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5. Solve $(D^3+16D)y=0$.

Given: $(D^3+16D)y=0$.

AE: $m^3+16m=0 \Rightarrow m(m^2+16)=0 \Rightarrow m=0, \; m^2=-16$

$m=\pm 4i$

G.S: $y=c_1e^{0x}+c_2\cos 4x+c_3\sin 4x$

6. Solve $(D^3+2D^2-8D)y=0$

Given: $(D^3+2D^2-8D)y=0$.

AE: $m^3+2m^2-8m=0$

$m(m^2+2m-8)=0 \Rightarrow m=0, \; m^2+2m-8=0$

$m^2+4m-2m-8=0$

$m(m+4)-2(m+4)=0$

$(m-2)(m+4)=0$

$m=2,-4$.

$\therefore m=0,-4,2$.

GS: $y=c_1e^{0x}+c_2e^{2x}+c_3e^{-4x}$.

7. Solve $(D^4-81)y=0$

Given: $(D^4-81)y=0$

A.E: $m^4-81=0$.

$\Rightarrow (m^2)^2 - (9)^2 = 0 \Rightarrow (m^2+9)(m^2-9) = 0.$

$m^2 = -9, \quad m^2 = 9$

$m = \pm 3i, \quad m = \pm 3$

G.S: $y = e^{0x}(c_1 \cos 3x + c_2 \sin 3x) + c_3 e^{3x} + c_4 e^{-3x}.$

8. Find the particular integral of $(D^2+4D+4)y = e^{-2x}$

Given: $(D^2+4D+4)y = e^{-2x}$

$PI = y_p = \frac{1}{f(a)} e^{ax}, \quad f(a) = a^2+4a+4$

$= \frac{1}{(-2)^2+4(-2)+4} e^{-2x}$

$= \frac{1}{8-8} e^{-2x}$ (Failure Case)

here $f(-2) = 0$ & $f'(a) = 2a+4$

$y_p = \frac{xe^{-2x}}{2(-2)+4}$ (Failure Case)

here $f'(-2) = 0$ & $f''(a) = 2$

$\therefore y_p = \frac{x^2 e^{-2x}}{2}.$

9. Find the particular integral of $(D^3+16D)y = \sin 4x.$

Given: $(D^3+16D)y = \sin 4x.$

P.I: $y_p = \frac{1}{f(D)} \sin ax = \frac{1}{D^3+16D} \sin 4x$

$= \frac{1}{D.D^2+16D} \sin 4x$

$= \frac{1}{-16D+16D} \sin 4x. \quad (D^2 = -a^2 = -4^2 = -16)$

$= \frac{x \sin 4x}{3D^2+16}$

$= \frac{x \sin 4x}{-48+16}$

$= -\frac{x \sin 4x}{32}.$

10. Find the particular integral of $\frac{d^3y}{dx^3} - y = (e^x+1)^2$

Given: $D^3y - y = (e^x+1)^2 \Rightarrow (D^3-1)y = (e^x+1)^2 = e^{2x}+1+2e^x$

$PI: y_p = \frac{1}{f(a)} e^{ax}, \quad f(a) = a^3-1$

$$= \frac{1}{2^3-1} e^{2x} + \frac{1}{(-1)} e^{0x} + \frac{2}{1-1} e^{x}$$

here $f(1)=0$ & $f'(a) = 3a^2$

$$= \frac{1}{7} e^{2x} - 1 + \frac{2x e^{x}}{3}$$

11. Find the particular integral of $(D^2-4)y = \cos^2 x$.

Given: $(D^2-4)y = \cos^2 x = \frac{1+\cos 2x}{2}$

PI: $y_p = \frac{1}{f(a)} e^{ax} + \frac{1}{f(D)} \cos 2x$

$$= \frac{1}{2(-4)} e^{0x} + \frac{1}{2(-2^2-4)} \cos 2x$$

$$= -\frac{1}{8} - \frac{\cos 2x}{16}.$$

12. Find the particular integral of $(D^2-6D+9)y = 18+54x$

Given: $(D^2-6D+9)y = 18e^{0x} + 54x$

$$y_p = \frac{18}{D^2-6D+9} e^{0x} + \frac{54}{D^2-6D+9} \cdot x$$

$$= \frac{18}{9} + \frac{54x}{(D-3)^2}$$

$$= \frac{18}{9} + \frac{54}{(-3)^2}\left(1-\frac{D}{3}\right)^{-2} x$$

$$= \frac{18}{9} + \frac{54}{9}\left[1 + \frac{2D}{3} + \frac{3D^2}{9} + \frac{4D^3}{9} + \cdots\right] x$$

$$= \frac{18}{9} + \frac{54}{9}\left[x + \frac{2}{3}\right] = \frac{18}{9} + \frac{54x}{9} + \frac{18}{9} \times 2$$

$$= 6(x+1)$$

13. Find the Particular Integral of $(D^2-a^2)y = x^4$

Given: $(D^2-a^2)y = x^4$

PI: $y_p = \frac{1}{D^2-a^2} \cdot x^4 = \frac{1}{(-a^2)\left[1-\left(\frac{D}{a}\right)^2\right]} x^4$

$$= -\frac{1}{a^2}\left[1-\left(\frac{D}{a}\right)^2\right]^{-1} x^4$$

$$= -\frac{1}{a^2}\left[1 + \frac{D^2}{a^2} + \frac{D^4}{a^4} + \frac{D^6}{a^6} \cdots\right] x^4$$

$$= -\frac{1}{a^2}\left[x^4 + \frac{D}{a^2}(4x^3) + \frac{D^3}{a^4}(4x^3)\right]$$

$$= -\frac{1}{a^2}\left[x^4 + \frac{12x^2}{a^2} + \frac{24}{a^4}\right]$$

$$= -\frac{x^4}{a^2} - \frac{12a^2}{a^4} - \frac{24}{a^6}.$$

14. Determine whether the functions $x^2, \frac{1}{x^2}$ are linearly independent on $(0,\infty)$

Let $f_1 = x^2;\ f_2 = \frac{1}{x^2}$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x^2 & \frac{1}{x^2} \\ 2x & -\frac{2}{x^3} \end{vmatrix} = -\frac{2}{x} - \frac{2}{x}$$

$$= -\frac{4}{x} \neq 0\ \forall x \in (0,\infty).$$

$\therefore x^2; \frac{1}{x^2}$ are linearly independent on the interval $(0,\infty)$

15. If $y_1 = \frac{1}{x}$ is a solution of $x^2y''+4xy'+2y=0$, then find the general solution of it by reducing its order.

Given: $x^2y''+4xy'+2y=0$ —(1)

Comparing with $a_0\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=0$.

$a_0=x^2;\ a_1=4x,\ a_2=2y.$

Given: $y_1 = 1/x$ is a solution of (1)

Another solution is $y_2 = y_1(x)\cdot u(x) = \frac{u(x)}{x}$

Now, $P(x) = \frac{a_1}{a_0} = \frac{4}{x}$, $v(x) = \frac{1}{y_1^2} e^{-\int P dx} = x^2 e^{-4\log x} = \frac{1}{x^2}$

$u(x) = \int \frac{1}{x^2} dx = -\frac{1}{x}.$

$\therefore y_2 = -\frac{1}{x^2}$

From method of reduction of order,

GS: $y = Ay_1(x) + By_2(x) = \frac{A}{x} - \frac{B}{x^2}.$

LAQs:

1. Solve $(D^2-4D+3)y = 3e^x\cos 2x.$

Given: $(D^2-4D+3)y = 3e^x\cos 2x$

AE: $m^2-4m+3=0$
$\Rightarrow m^2-3m-m+3=0$
$\Rightarrow m(m-3)-(m-3)=0$
$\Rightarrow (m-1)(m-3)=0$
$\Rightarrow m=1,3$

CF: $y_c = c_1e^x + c_2e^{3x}.$

PI: $y_p = \frac{3}{D^2-4D+3}\cdot e^x\cos 2x.$

$$= \frac{3e^x}{D^2-4D+3}\cos 2x$$

$$= \frac{3e^x}{(D+3)^2-4(D+3)+3}\cos 2x$$

$$= \frac{3e^{x}}{D^2+9+6D-4D-9}\cos 2x = \frac{3e^{x}}{D^2+2D}\cos 2x$$

$$= \frac{3e^{x}}{-4+2D}\cos 2x \qquad = \frac{3}{-2}e^{3x}\frac{(2+D)}{4-(-4)}\cos 2x$$

$$= -\frac{3}{8}e^{x}(2\cos 2x - 2\sin 2x).$$

$$y = y_c + y_p = c_1e^{x} + c_2e^{3x} - \frac{3e^{3x}}{8}(2\cos 2x - 2\sin 2x)$$

2. Solve $(D^2+2D+1)y = xe^{-x}$

Given: $(D^2+2D+1)y = xe^{-x}$

AE: $m^2+2m+1=0 \Rightarrow m^2+m+m+1=0$

$m(m+1)+(m+1)=0$

$(m+1)(m+1)=0$

$\therefore m=-1,-1$

CF: $y_c = (c_1+xc_2)e^{-x}$

PI: $y_p = \frac{1}{f(D)}e^{ax}V \quad = \frac{e^{-x}}{(D+1)^2}x$

$$= \frac{e^{-x}}{(D+1-1)^2}x \quad = \frac{e^{-x}}{D^2}x$$

$$= e^{-x}\int \frac{x^2}{2}dx = e^{-x}\frac{x^3}{6}.$$

$$\therefore y = y_c + y_p = (c_1+xc_2)e^{-x} + e^{-x}\frac{x^3}{6}.$$

3. Solve $(D^2+4)y = x^2+1+\cos 2x.$

Given: $(D^2+4)y = x^2+e^{0x}+\cos 2x$

AE: $m^2+4=0 \Rightarrow m=\pm 2i$

CF: $y_c = c_1\cos 2x + c_2\sin 2x.$

PI: $y_p = \frac{1}{f(D)}(x^2+e^{0x}+\cos 2x)$

$$= \frac{1}{D^2+4}x^2 + \frac{1}{4}e^{0x} + \frac{1}{D^2+4}\cos 2x$$

$$= \frac{1}{4\left(1+\frac{D^2}{4}\right)}x^2 + \frac{1}{4} + \frac{1}{-4+4}\cos 2x$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2 + \frac{1}{4} + \frac{x}{2D}\cos 2x.$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{D^4}{16}\cdots\right]x^2 + \frac{1}{4} + \frac{x\sin 2x}{4}$$

$$= \frac{x^2}{4} - \frac{1}{8} + \frac{1}{4} + \frac{x}{4}\sin 2x$$

$$= \frac{x^2}{4} + \frac{1}{8} + \frac{x}{4}\sin 2x$$

$$\therefore y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{x^2}{4} + \frac{1}{8} + \frac{x}{4}\sin 2x.$$

4. Show that e^x, e^{2x}, e^{3x} are the fundamental solution of $y'''-6y''+11y'-6y=0$ on any interval I.

Given: $y'''-6y''+11y'-6y=0$ $\Rightarrow (D^3-6D^2+11D-6)y=0$.

AE: $m^3-6m^2+11m-6=0$.

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \vert\, 0 \end{array}$$

$\Rightarrow m^2-5m+6=0$

$\Rightarrow m^2-2m-3m+6=0$

$\Rightarrow m(m-2)-3(m-2)=0$.

$\Rightarrow (m-3)(m-2)=0$

$\therefore m=1,2,3$.

GS: $y=c_1e^x+c_2e^{2x}+c_3e^{3x}$

$\therefore$ e^x, e^{2x}, e^{3x} are fundamental solutions on any interval I.

---

5. Find the general solution of $y''+16y=32\sec 2x$ by method of variation of parameters.

Given: $y''+16y=32\sec 2x$.

$D^2y+16y=32\sec 2x$

$(D^2+16)y=32\sec 2x$.

AE: $m^2+16=0 \Rightarrow m=\pm 4i$

CF: $y_c=c_1\cos 4x+c_2\sin 4x$.

From method of variation of parameters,

$y_c=c_1u+c_2v$ where $u=\cos 4x$, $v=\sin 4x$

$u'=-4\sin 4x$, $v'=4\cos 4x$

$uv'-vu'=4\cos^2 4x+4\sin^2 4x$

$=4(\cos^2 4x+\sin^2 4x)$.

$=4$.

PI: $y_p=Au+Bv=A\cos 4x+B\sin 4x$

$A=-\int\frac{vR}{uv'-vu'}dx=-\int\frac{32}{8}\sin 4x\sec 2x\,dx$

$=-8\int\frac{2\sin 2x\cos 2x}{\cos x}dx \quad = -\frac{16}{2}(-\cos 2x)$

$=8\cos 2x$.

$B=\int\frac{uR}{uv'-vu'}dx=8\int\cos 4x\sec 2x\,dx=8\int\frac{\cos^2 2x}{\cos 2x}-\frac{\sin^2 2x}{\cos 2x}dx$

$=8\int(\cos 2x-\tan 2x\sec 2x)dx \quad = 4(\sin 2x-\sec 2x)$

$y_p = 8\cos 2x \cos 4x + 4\sin 2x \sin 4x - 4\sec 2x \sin 4x.$

$y = y_c + y_p = c_1\cos 4x + c_2 \sin 4x + 8\cos 2x\cos 4x + 4\sin 2x \sin 4x - 4\sec 2x \sin 4x.$

6. Find the general solution of $y''+y = \sec x$ by method of variation of parameters.

Given: $y''+y=\sec x \Rightarrow (D^2+1)y = \sec x.$

AE: $m^2+1=0 \Rightarrow m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x.$

From method of variation of parameters,

$y_c = c_1 u + c_2 v$ where, $u = \cos x$, $v = \sin x$

$u' = -\sin x$, $v' = \cos x$

$uv' - vu' = \cos^2 x + \sin^2 x = 1$

PI: $y_p = Au + Bv = A\cos x + B\sin x$

$A = -\int \frac{vR}{uv'-vu'}dx = -\int \frac{\sin x \sec x}{1} dx = -\int \tan x\, dx = -\log|\sec x|$

$B = \int \frac{uR}{uv'-vu'}dx = \int \cos x \sec x\, dx = x.$

$y_p = -\cos x \log|\sec x| + x\sin x.$

$\therefore y = c_1\cos x + c_2 \sin x - \cos x \log|\sec x| + x\sin x.$

7. Solve $(D^2+1)y = x\cos x$ by method of variation of parameters.

Given: $(D^2+1)y = x\cos x$

AE: $m^2+1=0 \Rightarrow m = \pm i$

CF: $y_c = c_1\cos x + c_2 \sin x$

From method of variation of parameters,

$y_c = c_1 u + c_2 v$ where $u = \cos x$, $v = \sin x$, $uv' - vu' = 1$

$u' = -\sin x$ $v' = \cos x$

$y_p = Au + Bv = A\cos x + B\sin x$

$A = -\int \frac{vR}{uv'-vu'}dx = -\int x\sin x \cos x\, dx = -\frac{1}{2}\int x \sin 2x\, dx$

$= -\frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - \int \left(-\frac{\cos 2x}{2}\right)dx\right] = -\frac{1}{2}\left(-\frac{x}{2}\cos 2x + \frac{\sin 2x}{4}\right)$

$= \frac{x}{4}\cos 2x - \frac{\sin 2x}{8}$

$B = \int \frac{uR}{uv'-vu'} = \int x\cos^2 x\, dx = \int \frac{x}{2} + \frac{x}{2}\cos 2x\, dx$

$B = \frac{x^2}{4} + \frac{1}{2}\left[\frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right] = \frac{x^2}{4} + \frac{1}{4}\left[x\sin 2x + \frac{\cos 2x}{2}\right]$

$= \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{\cos 2x}{8}$

$y_P = \left[\frac{x}{4}\cos 2x - \frac{\sin 2x}{8}\right]\cos x + \left[\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{\cos 2x}{8}\right]\sin x$

$\therefore y = c_1\cos x + c_2\sin x + \left[\frac{x}{4}\cos 2x - \frac{\sin 2x}{8}\right]\cos x +$
$\left[\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{\cos 2x}{8}\right]\sin x.$

---

8. Solve $y'' + 4y = 4\tan 2x$ by method of variation of parameters:

Given: $y'' + 4y = 4\tan 2x \Rightarrow (D^2+4)y = 4\tan 2x.$

AE: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$y_c = c_1\cos 2x + c_2\sin 2x$

From method of variation of parameters,

$y_c = c_1u + c_2v$ where $u = \cos 2x$, $v = \sin 2x$

$u' = -2\sin 2x$, $v' = 2\cos 2x$

PI: $y_p = Au + Bv = A\cos 2x + B\sin 2x$

$A = -\int \frac{vR}{uv' - vu'}\,dx = -\frac{4}{2}\int \sin 2x \tan 2x\,dx = -2\int \cos 2x \tan^2 2x\,dx$

$= -2\int \cos 2x(\sec^2 2x - 1)\,dx = -2\int \sec 2x - \cos 2x\,dx$

$= -\frac{2}{2}\log|\sec 2x + \tan 2x| + \frac{2}{2}\sin 2x$

$B = \int \frac{uR}{uv' - vu'}\,dx = \frac{4}{2}\int \cos 2x \tan 2x\,dx = 2\int \sin 2x\,dx$

$= -\cos 2x.$

$y_p = [\sin 2x - \log|\sec 2x + \tan 2x|]\cos 2x - \cos 2x\sin 2x.$

$y = c_1\cos 2x + c_2\sin 2x + [\sin 2x - \log|\sec 2x + \tan 2x|]\cos 2x$
$- \cos 2x\sin 2x.$

---

9. Solve $(x^2D^2 - xD - 3)y = x^2\log x$

Given: $(x^2D^2 - xD - 3)y = x^2\log x$ —(1)

Clearly (1) is Euler-Cauchy's eqn

let $x = e^z \Rightarrow z = \log x$

so that $x\frac{dy}{dx} = Dy$ where $\frac{d}{dz} = D$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

From (1), $D(D-1)y - Dy - 3y = e^{2z}\cdot z$

$(D^2 - 2D - 3)y = e^{2z} z$

AE: $m^2 - 2m - 3 = 0 \Rightarrow m^2 - 3m + m - 3 = 0$

$m(m-3) + (m-3) = 0$

$(m+1)(m-3) = 0$

$m = -1, 3.$

$y_c = c_1 e^{-z} + c_2 e^{-3z} = \frac{c_1}{x} + \frac{c_2}{x^3}$

PI: $y_p = \frac{1}{f(D)} e^{az} V$

$= e^{az} \frac{1}{f(D+a)} V = \frac{e^{2z}}{(D+2)^2 - 2(D+2) - 3} z$

$= e^{2z} \frac{1}{D^2 + 4 + 4D - 2D - 4 - 3} z$

$= \frac{e^{2z}}{D^2 + 2D - 3} z.$

$= \frac{e^{2z}}{-3}\left[1 - \left(\frac{D^2}{3} + \frac{2D}{3}\right)\right]^{-1}$

$= \frac{e^{2z}}{-3}\left[1 + \frac{D^2}{3} + \frac{2D}{3} + \frac{D^4}{9} + \frac{4D^2}{9} + \frac{4}{9}D^3\right] z$

$= \frac{e^{2z}}{-3}\left[z + \frac{2}{3}\right] = \frac{e^{2z}}{-9}(3z + 2)$

$= \frac{x^2}{-9}(3\log x + 2)$

$\therefore y = c_1/x + c_2/x^3 + \frac{x^2}{(-9)}(3\log x + 2)$

---

10. Solve $(x^2 D^2 - xD - 1)y = x^3$

Given: $(x^2 D^2 - xD - 1)y = x^3$ —(1)

Clearly eqn (1) is EulerCauchy's eqn.

Let $x = e^z \Rightarrow z = \log x$

so that $x\frac{dy}{dx} = Dy$ & $x^2 \frac{d^2y}{dx^2} = D(D-1)y$ where $D = \frac{d}{dz}$

From (1), $(D^2 - D)y + Dy - y = e^{3z}$

$(D^2 - 1)y = e^{3z}$

AE: $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y_c = c_1 e^x + c_2 e^{-x}$

PI: $y_p = \frac{1}{f(a)} e^{az} = \frac{1}{9-1} e^{3z} = \frac{1}{8} x^3$

$y = c_1 e^x + c_2 e^{-x} + \frac{1}{8} x^3.$

11. Find the general solution of $x^3y''' - 3xy' + 3y = 16x + 9x^2\log x$

Given: $x^3y''' - 3xy' + 3y = 16x + 9x^2\log x$ — (1)

Clearly (1) is Euler Cauchy's eqn

Let $x = e^z \Rightarrow z = \log x$ so that $x\frac{dy}{dx} = Dy$ &

$x^2\frac{d^2y}{dx^2} = D(D-1)(D-2)y$.

From (1), $(D(D-1)(D-2) - 3D + 3)y = 16e^z + 9e^{2z}\cdot z$

$(D^3 - 3D^2 - D + 3)y = 16e^z + 9e^{2z}\cdot z$

AE: $m^3 - 3m^2 - m + 3 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -3 & -1 & 3 \\ & 0 & 1 & -2 & -3 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

$\Rightarrow m^2 - 2m - 3 = 0 \Rightarrow m^2 - 3m + m - 3 = 0$

$m(m-3) + (m-3) = 0$

$(m+1)(m-3) = 0$

$m = 1, -1, 3.$

$y_c = c_1e^x + c_2e^{-x} + c_3e^{3x}$

$y_p = \frac{1}{f(D)} 16e^z + 9e^{2z}\cdot z = \frac{16e^z}{1-3+3-1} + \frac{9e^{2z}}{(D+2)^3 - 3(D+2)^2 - (D+3) + 3} z.$

$= \frac{16ze^z}{3-6-1} + \frac{9e^{2z}}{D^3+8+6D^2+12D-3D^2-12-12D-D-2+3}\cdot z.$

$= -4ze^z + \frac{9e^{2z}}{D^3+3D^2-D-3}\cdot z$

$= -4ze^z - 3e^{2z}(z - \frac{1}{3})$

$= -4x\log x - 3x^2(\log x - \frac{1}{3})$

$y = c_1e^x + c_2e^{-x} + c_3e^{3x} - 4x\log x - 3x^2(\log x - \frac{1}{3}).$

---

12. Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$

Given: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$ — (1)

Clearly equ(1) is Euler Cauchy's eqn

let $x = e^z$, $z = \log x$ then $x\frac{dy}{dx} = Dy$ & $x^2\frac{d^2y}{dx^2} = D(D-1)(D-2)y$

From (1), $D(D-1)y + Dy - 4y = e^{2z}$

$(D^2 - 4)y = e^{2z}$

AE: $m^2 - 4 = 0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$

$y_c = c_1e^{2z} + c_2e^{-2z} = c_1x^2 + \frac{c_2}{x^2}$

PI : $y_p = \frac{1}{f(a)} e^{az} = \frac{1}{4-4} e^{2z}$

$= \frac{ze^{2z}}{2D} = \frac{ze^{2z}}{4}$

$= \frac{x^2 \log x}{4}$

$\therefore y = c_1 e^{2z} + c_2 e^{-2z} + \frac{z}{4} e^{2z}$

$y = c_1 x^2 + \frac{c_2}{x^2} + \frac{x^2}{4} \log x.$

13. If $y_1 = e^{2x}$ is a solution of $y'' - 5y' + 6y = 0$, find second linearly independent solution.

Given: $y'' - 5y' + 6y = 0$ —(1)

Compare with $a_0 y'' + a_1 y' + a_2 y = 0$.

$a_0 = 1, \quad a_1 = -5, \quad a_2 = 6.$

Given: $y_1 = e^{2x}$ is a solution of (1).

Another solution is $y_2 = y_1(x) u(x) = e^{2x} u(x)$

$P(x) = \frac{a_1}{a_0} = -5, \quad v(x) = \frac{1}{y_1^2} e^{-\int P dx} = e^{-4x} e^{5x} = e^x$

$u(x) = \int v(x)\, dx = \int e^x dx = e^x.$

$y_2 = e^{2x} e^x = e^{3x}$

From method of reduction of order,

$y = A y_1(x) + B y_2(x) = A e^{2x} + B e^{3x}$

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Unit - IV

Special functions

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SAQs :-

D-17-R (1) Evaluate $\int_0^\infty x^{1/3} e^{-x^2} dx$

D-17-B (2) Evaluate $\int_0^\infty x^4 e^{-x^4} dx$ in terms of Gamma function.

J-17-R (3) Evaluate $\int_0^\infty t^4 e^{-2t^2} dt$.

M-19-R (4) Evaluate the improper integral $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$.

J-16-B (5) Evaluate $\int_0^\infty e^{-x^2} dx$.

D-17-R (6) Evaluate $\int_0^n x^n \left(1-\frac{x}{m}\right)^{m-1} dx$ in terms of beta function.

D-17-B (7) Evaluate $\int_0^1 x^p (1-x^q)^r dx$ in terms of β-function, where p, q, r are +ve integers.

D-19-B (8) Using Beta and Gamma functions evaluate the integral $\int_{-1}^{1} (1-x^2)^n dx$ where 'n' is a +ve integer.

D-16-R (9) Evaluate $\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}} dx$ using Beta and Gamma function.

J-15-R (10) Evaluate $\int_a^b (x-a)^{m-1} (b-x)^{n-1} dx$ interms of β-function where m, n, a, b are +ve constants.

J-15-R (11) Define error function and prove that $\operatorname{erf}(-x) = -\operatorname{erf}(x)$.

J-15-R (12) Evaluate $\frac{d}{dx}[\operatorname{erfc}(\alpha x)]$

J-17-B (13) Evaluate $\frac{d}{dx}\{\operatorname{erf}(\alpha x)\}$

A-16-R
J-16-B

J-15-R
(14) Using Rodrigue's formula find $P_2(x)$.

D-17-R
(15) Evaluate $4P_3(x) + 6P_2(x) + 3P_1(x)$ as a polynomial of 'x'.

D-19-B
(16) Express $3P_3(x) + 2P_2(x) + 4P_1(x) + 5P_0(x)$ as a polynomial in 'x' where $P_n(x)$ is the Legendre's polynomial of order 'n'.

J-17-R
(17) Express $f(x) = 5x^3 + 6x^2 + 4$ interms of Legendre's polynomials.

LAQs :-

D-16-B
(1) Prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$.

A-16-R
(2) Show that $\beta(m, \frac{1}{2}) = 2^{2m-1}\,\beta(m,n)$

J-16-B
(3) Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

J-17-R
(4) Evaluate $\int_0^\infty e^{-mx}(1-e^{-x})^n\, dx$ where m, n are +ve integers.

D-16-R
(5) Evaluate $\int_0^{\pi/2} \sin^5\theta \cos^7\theta\, d\theta$ using beta and gamma functions.

J-16-B
(6) Prove that $\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}$.

A-16-R
(7) Show that $\Gamma\frac{1}{2} = \sqrt{\pi}$

D-16-B, J-16-old
(8) Find the power series solution of
$$(1-x^2)y'' - 2xy' + 2y = 0.$$

J-16-B
(9) Find the series solution of $y'' + x^2 y = 0$ about $x = 0$.

D-17-B
(10) Show that $erf(-x) + erf(x) = 0$.

J-17-R
(11) Show that $\int_0^t erf(ax)\,dx = t\,erf(at) + \frac{1}{a\sqrt{\pi}}\left[e^{-a^2t^2} - 1\right]$

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UNIT-IV : Special Functions

Solutions

1. Evaluate $\int_0^\infty e^{-x^2} x^{1/3}\, dx$.

Let $x^2 = t \Rightarrow x = \sqrt{t}$

$dx = \frac{dt}{2\sqrt{t}}$

$$\int_0^\infty e^{-x^2} x^{1/3} dx = \int_0^\infty e^{-t} t^{1/6} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty e^{-t} t^{-1/3} dt$$

We know, $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$.

here, $n-1 = -\frac{1}{3} \Rightarrow n = \frac{2}{3}$.

$\Gamma n = \frac{1}{2}\Gamma\frac{2}{3}$

2. Evaluate $\int_0^\infty e^{-x^4} x^4\, dx$ in terms of Gamma Function

Let $x^4 = t \Rightarrow x = t^{1/4}$

$dx = \frac{1}{4} t^{-3/4} dt$

$$\int_0^\infty e^{-x^4} x^4 dx = \int_0^\infty e^{-t} \frac{t}{4} t^{-3/4} dt = \frac{1}{4}\int_0^\infty e^{-t} t^{1/4} dt$$

here, $n-1 = \frac{1}{4} \Rightarrow n = \frac{5}{4}$

$= \frac{1}{4}\Gamma n$

$= \frac{1}{4}\Gamma\frac{5}{4}$

3. Evaluate $\int_0^\infty t^4 e^{-2t^2} dt$.

let $x = 2t^2 \Rightarrow t = \frac{\sqrt{x}}{\sqrt{2}}$

$dt = \frac{1}{2\sqrt{2}\sqrt{x}} dx$

$$\int_0^\infty t^4 e^{-2t^2} dt = \int_0^\infty e^{-x} \frac{x^{4/2}}{2^{4/2}} \frac{x^{-1/2}}{2\sqrt{2}} dx = \frac{1}{8\sqrt{2}}\int_0^\infty e^{-x} x^{3/2} dx$$

here, $n-1 = \frac{3}{2} \Rightarrow n = \frac{5}{2}$

$= \frac{1}{8\sqrt{2}}\Gamma n = \frac{1}{8\sqrt{2}}\Gamma\frac{5}{2}$

$= \frac{1}{8\sqrt{2}}\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right) = \frac{3}{16\sqrt{2}}\frac{1}{2}\Gamma\frac{1}{2}$

$= \frac{3}{32\sqrt{2}}\sqrt{\pi}$

4. Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$

Let $x^2 = t \Rightarrow x = \sqrt{t}$
$dx = \frac{dt}{2\sqrt{t}}$

$$\int_0^\infty \sqrt{x} e^{-x^2} dx = \int_0^\infty t^{1/4} e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty e^{-t} t^{-1/4} dt.$$

here, $n-1 = -\frac{1}{4} \Rightarrow n = \frac{3}{4}$

$$= \frac{1}{2}\Gamma n = \frac{1}{2}\Gamma\frac{3}{4}$$

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5. Evaluate $\int_0^\infty e^{-x^2} dx$

let $x^2 = t \Rightarrow x = \sqrt{t}$
$dx = \frac{dt}{2\sqrt{t}}.$

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} dt.$$

$n-1 = -\frac{1}{2}$
$n = 1-\frac{1}{2}$
$n = \frac{1}{2}.$

$$= \frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2}$$

6. Evaluate $\int_0^n x^n (1-\frac{x}{m})^{m-1} dx$ in terms of β-function.

Given: $\int_0^n x^n (1-\frac{x}{m})^{m-1} dx$

Let $\frac{x}{m} = y \Rightarrow dx = m\,dy.$

UL: $x \to n$ then $y \to \frac{n}{m}$, L.L: $x \to 0$ then $y \to 0$.

$$\int_0^n x^n (1-\frac{x}{m})^{m-1} dx = \int_0^{n/m} (1-y)^{m-1} (my)^n m\, dy.$$

$$= \int_0^{n/m} (1-y)^{m-1} m^{n+1} y^n dy.$$

Let $m = n$,
& $\frac{n}{m} = 1$

$$= \int_0^1 (1-y)^{n-1} n^{n+1} y^{(n+1)-1} dy.$$

$$= n^{n+1}\int_0^1 (1-y)^{n-1} y^{(n+1)-1} dy.$$

We know $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

$$\int_0^n x^n (1-\frac{x}{m})^{m-1} dx = n^{n+1}\beta(n+1, n)$$

7. Evaluate $\int_0^1 x^p(1-x^q)^r dx$ in terms of β-function, where p, q, r are +ve integers:

Given: $\int_0^1 x^p(1-x^2)^r dx$

Let $x^q = t \Rightarrow x = t^{1/q}$

$dx = \frac{1}{q} t^{\frac{1}{q}-1} dt$

$= \int_0^1 t^{p/q}(1-t)^r \frac{t}{q}^{\frac{1}{q}-1} dt$

$= \frac{1}{q}\int_0^1 (1-t)^r t^{\frac{p}{q}+\frac{1}{q}-1} dt$ $\quad (\because \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx)$

$= \frac{1}{q}\left(\frac{p}{q}+\frac{1}{q}, r+1\right)$

---

8. Using β & γ functions, evaluate the integral $\int_{-1}^1 (1-x^2)^n dx$ where 'n' is a +ve integer.

Given: $\int_{-1}^1 (1-x^2)^n dx$

Let $I = \int_{-1}^1 (1-x^2)^n dt = \int_{-1}^1 (1-x)^n(1+x)^n dx$

Let $(1+x) = 2y \Rightarrow dx = 2dy$. , U.L: $y \to 1$, LL: $y \to 0$.

$I = \int_0^1 (1-(2y-1))^n (2y)^n \, 2dy$.

$= 2\int_0^1 2^n y^n [2^n(1-y)^n] dy$.

$= 2^{2n+1}\int_0^1 y^{(n+1)-1}(1-y)^{(n+1)-1} dy$.

$= 2^{2n+1} \beta(n+1, n+1)$

$= 2^{2n+1} \frac{\overline{|n+1}\,\overline{|n+1}}{\overline{|2(n+1)}}$.

We know, $\beta(m,n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$ &

$\overline{|n+1} = n!$

$= 2^{2n+1} \frac{n!\,n!}{(2n+1)!}$

$= 2^{2n+1} \frac{(n!)^2}{(2n+1)!}$

9. Evaluate $\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\,dx$ using β & γ functions.

$$\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\,dx = \int_0^{\pi/2} \sin^{-1/2} x \cos^0 x\,dx$$

We know, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$.

$2m-1 = -\frac{1}{2}$

$2m = \frac{1}{2}$

$m = \frac{1}{4}$

$2n-1 = 0$

$n = \frac{1}{2}$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}$$

$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma 1/4}{\Gamma 3/4}.$$

---

10. Evaluate $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$ interms of β-function

We know, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$.

Let $x = \frac{t-a}{b-a} \Rightarrow dx = \frac{dt}{b-a}$

U.L: $x \to 1$ then $t \to b$, L.L: $x \to 0$ then $t \to a$.

$$\int_a^b \frac{(t-a)^{m-1}(b-a-t+a)^{n-1}\,dt}{(b-a)^{m-1+n-1+1}}$$

$$\beta(m,n) = \frac{1}{(b-a)^{m+n-1}}\int_a^b (t-a)^{m-1}(b-t)^{n-1}\,dt.$$

$$\therefore (b-a)^{m+n-1}\beta(m,n) = \int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$$

---

11. Define error function & prove that $erf(-x) = -erf(x)$

ERROR-FUNCTION: Error function is defined as $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

As we know, $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

Put $t = -y \Rightarrow dy = -dt$.

$$erf(x) = -\frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-y^2}\,dy.$$

$$= -\frac{2}{\sqrt{\pi}}\,erf(-x)$$

$$\therefore erf(-x) = -erf(x).$$

12. Evaluate $\frac{d}{dx}[\operatorname{erfc}(\alpha x)]$

We know, $\frac{d}{dx}\operatorname{erf}(\alpha x) = \frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}$.

Now, $\operatorname{erfc}(\alpha x) = 1 - \operatorname{erf}(\alpha x)$

$$\frac{d}{dx}\operatorname{erfc}(\alpha x) = -\frac{d}{dx}\operatorname{erf}(\alpha x)$$

$$= -\frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}.$$

13. Evaluate $\frac{d}{dx}\operatorname{erf}(\alpha x) =$

We know, $\frac{d}{dx}\int_0^x f(y)\,dy = f(x)$

Now, $\frac{d}{dx}\operatorname{erf}(\alpha x) = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{\alpha x} e^{-t^2}dt$

$$= \frac{2}{\sqrt{\pi}} e^{-\alpha^2 x^2}\frac{d}{dx}\alpha x$$

$$= \frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}.$$

14. Using Rodrigue's formula, find $P_2(x)$

Rodrigue's formula is $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

Let $n=2$, $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2$.

$$= \frac{1}{8}\frac{d}{dx}2(x^2-1)(2x)$$

$$= \frac{1}{2}\frac{d}{dx}(x^3-x)$$

$$= \frac{1}{2}(3x^2-1)$$

15. Evaluate $4P_3(x)+6P_2(x)+3P_1(x)$ as a polynomial of x

We know, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$ & $P_3(x) = \frac{1}{2}(5x^3-3x$

$$4P_3(x)+6P_2(x)+3P_1(x) = \frac{6}{2}(3x^2-1)+\frac{4}{2}(5x^3-3x)+3x$$

$$= 9x^2-3+10x^3-6x+3x$$

$$= 10x^3+9x^2-3x-3.$$

16. Express $3P_3(x)+2P_2(x)+4P_1(x)+5P_0(x)$ as a polynomial in 'x' when $P_n(x)$ is Legendre's polynomial of order n.

We know, $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$.

$P_3(x)=\frac{1}{2}(5x^3-3x)$

$$\therefore P(x)=\frac{3}{2}(5x^3-3x)+\frac{2}{2}(3x^2-1)+4x+5$$

$$=\frac{15}{2}x^3-\frac{9}{2}x+3x^2-1+4x+5$$

$$=\frac{15}{2}x^3+3x^2-\frac{x}{2}+4$$

$$=\frac{1}{2}[15x^3+6x^2-x+8]$$

---

17. Express $f(x)=5x^3+6x^2+4$ in terms of Legendre's polynomial

Given: $f(x)=5x^3+6x^2+4$.

We know, $P_0(x)=1$, $P_1(x)=x$.

$P_2(x)=\frac{1}{2}(3x^2-1) \Rightarrow x^2=\frac{2P_2(x)+1}{3}=\frac{2}{3}P_2(x)+\frac{P_0(x)}{3}$

$P_3(x)=\frac{1}{2}(5x^3-3x) \Rightarrow x^3=\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)$

$$\therefore f(x)=2P_3(x)+3P_1(x)+4P_2(x)+2P_0(x)+4P_0(x)$$

$$=2P_3(x)+4P_2(x)+3P_1(x)+6P_0(x)$$

LAQ 8:

1. Prove that $\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

We know, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $x=\frac{1}{1+y} \Rightarrow dx=-\frac{1}{(1+y)^2}dy$

L.L: $y=\infty$, U.L: $y=0$.

$$\therefore \beta(m,n)=\int_\infty^0\left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}dy\right)$$

$$=\int_0^\infty\frac{1}{(1+y)^{m+1}}\frac{y^{n-1}}{(1+y)^{n-1}}dy$$

$$=\int_0^\infty\frac{y^{n-1}}{(1+y)^{m+n}}dy$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

We know, from symmetry of β-function

$$\beta(m,n) = \beta(n,m)$$

$$\therefore\ \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

2. Show that $\beta(m,\tfrac{1}{2}) = 2^{2m-1}\beta(m,m)$

We have, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$ —— ①

Put $n = 1/2$ then,

$$\beta(m,\tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta \quad \text{—— ②}$$

Put $n = m$ in (1),

$$\beta(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta.$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} [2\sin\theta\cos\theta]^{2m-1}\, d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\, d\theta$$

Let $2\theta = \phi \Rightarrow d\theta = \frac{d\phi}{2}$

$$\beta(m,m) = \frac{2}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\, \frac{d\phi}{2}$$

$$2^{2m-1}\beta(m,m) = \int_0^{\pi} \sin^{2m-1}\phi\, d\phi$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi$$

$$= \beta(m,\tfrac{1}{2}) \qquad (\because \text{From } (2))$$

$$\therefore\ \beta(m,\tfrac{1}{2}) = 2^{2m-1}\beta(m,m).$$

3. Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$$\int_0^{\pi/2} \tan^{1/2}\theta\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta.$$

We know, $\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$

$2m-1 = \frac{1}{2} \Rightarrow m = \frac{3}{4}$ & $2n-1 = -\frac{1}{2} \Rightarrow n = \frac{1}{2}$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2}\frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}$$

$$= \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$

$$= \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) \qquad \left(\because \Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}\right)$$

$$= \frac{1}{2}\frac{\pi}{\sin \pi/4}$$

$$= \frac{\pi}{2}\sqrt{2}$$

$$\therefore \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$$

---

4. Evaluate $\int_0^\infty e^{-mx}(1-e^{-x})^n dx$ where m, n are +ve integers.

Put $e^{-x} = t$.

$$-e^{-x}dx = dt$$

$$dx = -\frac{dt}{t}$$

L.L: $x \to 0$ then $t \to 1$, U.L: $x \to \infty$ then $t \to 0$.

$$\int_0^\infty e^{-mx}(1-e^{-x})^n dx = \int_0^1 t^m (1-t)^n \frac{dt}{t}$$

$$= \int_0^1 t^{m-1}(1-t)^{(n+1)-1} dt$$

$$= \beta(m, n+1).$$

---

5. Evaluate $\int_0^{\pi/2} \sin^5\theta \cos^7\theta\, d\theta$ using β & γ functions

We know, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$.

$2m-1 = 5$ $2n-1 = 7$

$m = 3$ $n = 4$

$$\int_0^{\pi/2} \sin^5\theta \cos^7\theta\, d\theta = \frac{1}{2}\beta(3,4)$$

$$= \frac{1}{2}\frac{\Gamma 3\,\Gamma 4}{\Gamma 7} \qquad \left[\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right]$$

$$= \frac{1}{2}\frac{3!\,2!}{6!}$$

$$= \frac{6}{720}$$

$$= 1/120$$

6. Prove that $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

Statement: If $m>0$ & $n>0$ then $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

Proof: We know, $\Gamma m = \int_0^\infty e^{-x} x^{m-1} dx$

let $x = u^2$, $dx = 2u\,du$

$= \int_0^\infty e^{-u^2} u^{2m-2} 2u\,du$

$= 2\int_0^\infty u^{2m-1} e^{-u^2} du$ —— (1)

$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$ let $x = v^2$, $dx = 2v\,dv$

$= 2\int_0^\infty v^{2n-1} e^{-v^2} dv$ —— (2)

From (1) & (2), $\Gamma m \Gamma n = 4\int_0^\infty\int_0^\infty u^{2m-1} v^{2n-1} e^{-(u^2+v^2)} du\,dv$ —— (3)

Changing to polar coordinates

$u = r\cos\theta$, $v = r\sin\theta$, $du\,dv = r\,dr\,d\theta$, $u^2+v^2 = r^2$

Limits for 'r': If $u\to 0$ & $v\to 0$ then $r\to 0$

If $u\to\infty$ & $v\to\infty$ then $r\to\infty$.

Limits for 'θ': $\frac{u}{v} = \tan\theta$

If $u\to 0$ & $v\to 0$ then $\theta\to 0$

If $u\to\infty$ & $v\to\infty$ then $\theta\to \pi/2$

From (3), $\Gamma m \Gamma n = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} r^{2m-1+2n-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, e^{-r^2} r\,dr\,d\theta$

$\Gamma m \Gamma n = \left[2\int_{r=0}^{\infty} r^{2m+2n-1} e^{-r^2} dr\right]\left[2\int_{\theta=0}^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right]$ —— (4)

We know, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta$

From (4), $\Gamma m \Gamma n = \beta(m,n)\left[2\int_{r=0}^{\infty} r^{2m+2n-1} e^{-r^2} dr\right]$ —— (5)

Now, $\Gamma m+n = \int_0^\infty e^{-x} x^{m+n-1} dx$

Put $x = r^2 \Rightarrow dx = 2r\,dr$

$\Gamma m+n = \int_0^\infty e^{-r^2} r^{2m+2n-2} 2r\,dr$

$= 2\int_0^\infty r^{2m+2n-1} e^{-r^2} dr$

From (5), $\Gamma m \Gamma n = \beta(m,n)\Gamma m+n \Rightarrow \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$.

7. Show that $\Gamma\frac{1}{2} = \sqrt{\pi}$

Proof: We know, $\beta(m,n) = \dfrac{\Gamma m \Gamma n}{\Gamma m+n}$

Let $m=n=\frac{1}{2}$ then $\beta(\frac{1}{2},\frac{1}{2}) = (\Gamma\frac{1}{2})^2$ —— ①

$$\beta(m,n) = \beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^1 x^{1/2-1}(1-x)^{1/2-1}dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}dx$$

Let $x = \sin^2\theta \Rightarrow dx = \sin 2\theta\, d\theta$

U.L: $x\to 1 \Rightarrow \theta \to \pi/2$, L.L: $x\to 0 \Rightarrow \theta\to 0$.

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^{\pi/2} \sin^{-1}\theta(1-\sin^2\theta)^{-1/2} 2\sin\theta\cos\theta\, d\theta.$$

$$= 2\int_0^{\pi/2} d\theta = 2\,\theta\Big]_0^{\pi/2} = \pi.$$

$\beta(\frac{1}{2},\frac{1}{2}) = \pi$ —— ②

$(1) = (2) \Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi$

$\therefore \Gamma\frac{1}{2} = \sqrt{\pi}$

---

8. Find the power series solution of $(1-x^2)y'' - 2xy' + 2y = 0$

Given: $(1-x^2)y'' - 2xy' + 2y = 0$ —— ①.

At $x=0$, $1-x^2 \neq 0$.

$\therefore x=0$ is an ordinary point of (1).

Let $y = a_0 + a_1x + a_2x^2 + \ldots\ldots = \sum_{n=1}^{\infty} a_n x^n$ —— ②

$y' = a_1 + 2a_2x + \ldots\ldots = \sum_{n=1}^{\infty} n a_n x^{n-1}$ —— ③

$y'' = 2a_2 + 6a_3x + \ldots\ldots = \sum_{n=1}^{\infty} n(n-1)a_n x^{n-2}$ —— ④

Sub^{n} (2), (3), (4) in (1).

$(1-x)^2[2a_2 + 6a_3x + 12a_4x^2 + \ldots] - 2x[a_1 + 2a_2x + 3a_3x^2 + \ldots] + 2[a_0 + a_1x + a_2x^2 + \ldots] = 0$

Comp coeff of – $x^0 \Rightarrow 2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$

$x \Rightarrow 6a_3 - 2a_1 + 2a_1 = 0 \Rightarrow a_3 = 0.$

$x^2 \Rightarrow -2a_2 - 4a_2 + 2a_2 + 12a_4 = 0 \Rightarrow a_4 = \frac{-a_0}{3}$

$x^3 \Rightarrow -6a_3 - 6a_3 + 2a_3 = 0 \Rightarrow a_3 = 0$

Sol^n is $y = a_0 + a_1x - a_0x^2 - \frac{a_0}{3}x^4 + \ldots = a_0\left[1 - x^2 - \frac{x^4}{3}\ldots\right]$

where a_0 & a_1 are arbitrary constants.

9. Find the series solution of $y''+x^2y=0$ about $x=0$.

Given: $y''+x^2y=0$ —— (1)

here, $P_0(x)=1\neq 0$ for $x=0$.

$\therefore$ $x=0$ is an ordinary point of (1).

Let $y=a_0+a_1x+a_2x^2+a_3x^3+\dots = \sum_{n=0}^{\infty} a_n x^n$ —— (2)

$y'=a_1+2a_2x+3a_3x^2+\dots = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$y''=2a_2+6a_3x+12a_4x^2+\dots = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$ —— (3)

Subn (2), (3) in (1),

$$2a_2+6a_3x+12a_4x^2+20a_5x^3+\dots+x^2a_0+a_1x^3+a_2x^4+\dots=0.$$

Comp coeff of – $x^0 \Rightarrow 2a_2=0 \Rightarrow a_2=0$

$x \Rightarrow 6a_3=0 \Rightarrow a_3=0$

$x^2 \Rightarrow 12a_4+a_0=0 \Rightarrow a_4=\frac{-a_0}{12}$.

$x^3 \Rightarrow 20a_5+a_1=0 \Rightarrow a_5=\frac{-a_1}{20}$.

$$y=a_0+a_1x-\frac{a_0}{12}x^4-\frac{a_1}{20}x^5.$$

$$=a_0\left[1-\frac{x^4}{12}\right]+a_1\left[x-\frac{x^5}{20}\right]$$

---

10. Show that $erf(-x)+erf(x)=0$.

We know, $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

Let $t=-y$, $dt=-dy$

$$=\frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-y^2}(-dy)$$

$$=-\frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-y^2}dy$$

$$=-erf(-x)$$

$$-erf(x)=erf(-x)$$

Given: $erf(-x)+erf(x)$

$=erf(-x)-erf(-x)$

$=0$.

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J-17-R

(68a)

(11) Show that $\int_0^t \operatorname{erf}(\alpha x)\,dx = t\operatorname{erf}(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}\left[e^{-\alpha^2 t^2} - 1\right]$

Sol:-

$$\int_0^t \underset{u}{\operatorname{erf}(\alpha x)} \cdot \underset{v}{1}\,dx = \left[\operatorname{erf}(\alpha x)\cdot x\right]_0^t - \int_0^t \frac{d}{dx}\operatorname{erf}(\alpha x)\cdot x\,dx$$

$$= t\operatorname{erf}(\alpha t) - \int_0^t \frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}\cdot x\,dx \qquad \boxed{\because \frac{d}{dx}\operatorname{erf}(\alpha x) = \frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}}$$

$$= t\operatorname{erf}(\alpha t) - \frac{2\alpha}{\sqrt{\pi}}\int_0^t e^{-\alpha^2 x^2}\cdot x\,dx$$

Now put $\alpha^2 x^2 = y \Rightarrow x^2 = \frac{y}{\alpha^2} \Rightarrow 2x\,dx = \frac{dy}{\alpha^2}$

$$\therefore \int_0^t \operatorname{erf}(\alpha x)\,dx = t\operatorname{erf}(\alpha t) - \frac{\alpha}{\sqrt{\pi}}\int_0^{\alpha^2 t^2} e^{-y}\,\frac{dy}{\alpha^2}$$

$$= t\operatorname{erf}(\alpha t) - \frac{1}{\alpha\sqrt{\pi}}\left[\frac{e^{-y}}{-1}\right]_0^{\alpha^2 t^2}$$

$$\int_0^t \operatorname{erf}(\alpha x)\,dx = t\operatorname{erf}(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}\left[e^{-\alpha^2 t^2} - 1\right]$$

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Unit V : Laplace Transforms

SAQs :-

M-19
(1) Find the Laplace transform of $f(t) = t^3$.

J-16
(2) Find $L\{\sin^2 t\}$.

J-15
(3) Find $L\{\cos(at+b)\}$ where a, b are constants.

D-19
(4) Find the Laplace transform of piece-wise continuous function
$$f(t) = \begin{cases} 0 & , \quad 0 \le t \le 2 \\ k & , \quad t \ge 2 \end{cases}$$
where k is a constant.

D-17 (R)
(5) Find $L\{e^{-2t}(\cos 3t - \sin 3t)\}$

J-17 (R)
(6) Find $L\{t^3 e^{-4t}\}$

D-17 (B)
(7) Evaluate $L\{e^{-t}(t+2)\}$

J-17 (B)
(8) Evaluate $L\{t^2 \cosh at\}$

J-15
(9) Find $L\left\{\frac{\sin t}{t}\right\}$

M-19
(10) State necessary and sufficient conditions for existence of Laplace transform.

M-19
(11) Find $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

D-17 (B)
(12) Evaluate $L^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\}$

J-17 (M)
(13) Find $L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$

A-16 (M)
(14) Evaluate $L^{-1}\left\{\frac{s}{s^2-4}\right\}$

J-17 (B)

(15) Evaluate $L^{-1}\left\{\dfrac{5s+10}{9s^2-16}\right\}$

D-17 (R)

(16) Find $L^{-1}\left\{\dfrac{3s+2}{(s+1)^3}\right\}$

D-16 (B)

(17) Find $L^{-1}\left\{\dfrac{e^{-2s}}{s^2-4}\right\}$

J-15

(18) Find $L^{-1}\left\{\dfrac{s+6}{s^2+6s+13}\right\}$

D-19

(19) Solve the initial value problem using Laplace transform $y''+4y=0$, $y(0)=1$, $y'(0)=6$.

LAQs :-

D-17 (M)

(1) Evaluate $L\left\{\dfrac{\cos at-\cos bt}{t}\right\}$

D-17 (B)

(2) Evaluate $L\left\{\int_0^t \dfrac{\sin 3u}{u}\,du\right\}$

J-17 (M)

(3) Evaluate $L\left\{e^{-t}\int_0^t \dfrac{\sin u}{u}\,du\right\}$

J-17 (B)

(4) Evaluate $L\left\{\int_0^t e^{u}\dfrac{\sin u}{u}\,du\right\}$

D-17 (M)

(5) Evaluate $L^{-1}\left\{\dfrac{s}{(s-3)(s^2+4)}\right\}$

D-19

(6) Find $L^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

D-17 (B)

(7) Evaluate $L^{-1}\left\{\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)\right\}$

J-17, J-16

(8) Evaluate $L^{-1}\left\{\log\left(\dfrac{s+3}{s+4}\right)\right\}$

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A-16(M)

(9) Find $L^{-1}\left\{\frac{e^{-as}}{s^2+1}+\frac{e^{-s}}{s^3}\right\}$

J-17 (B)

(10) Evaluate $L^{-1}\left\{\cot^{-1}\left(\frac{s+3}{2}\right)\right\}$

D-16

(11) Apply convolution theorem to find $L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$

J-15(M)

(12) Find the inverse Laplace transform of $F(s)=\frac{s}{(s^2+a^2)^2}$ by using convolution theorem.

D-17(M)

(13) Solve $y''+3y'+2y=3$, $y(0)=1$, $y'(0)=1$ by applying Laplace transform.

D-17(B)

(14) Solve $y''+y=2e^{t}$, $y(0)=0$, $y'(0)=2$, by using Laplace transform.

J-17(M)

(15) Solve $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=e^{t}$ where $y(0)=2$, $y'(0)=-1$ by method of Laplace transform.

A-16(M)

(16) Solve $\frac{d^2y}{dt^2}+9y=\cos 2t$ if $y(0)=1$, $y\left(\frac{\pi}{2}\right)=-1$ by the method of Laplace transform.

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UNIT 5- LAPLACE TRANSFORM.

SAQs:

1. Find the Laplace transform of $f(t)=t^3$.

Given: $f(t)=t^3$

$$L\{f(t)\} = L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4} \qquad \left(\because L\{t^n\} = \frac{n!}{s^{n+1}}\right)$$

2. Find $L\{\sin^2 t\}$

Given: $L\{f(t)\} = L\{\sin^2 t\}$

$$= L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= L\left\{\frac{1}{2}\right\} - L\left\{\frac{\cos 2t}{2}\right\}$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+4)} \qquad \left(\because L\{\cos at\} = \frac{s}{s^2+a^2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{(s^2+4)}\right]$$

3. Find $L\{\cos(at+b)\}$ where a,b are constants.

We know that $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

$$L\{\cos(at+b)\} = L\{\cos at\cos b\} - L\{\sin at\sin b\}$$

$$= \cos b\left[\frac{s}{s^2+a^2}\right] - \sin b\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{s^2+a^2}(s\cos b - a\sin b)$$

4. Find the Laplace transform of piece-wise continuous function $f(t) = \begin{cases} 0, & 0 \le t \le 2 \\ k, & t \ge 2 \end{cases}$ where k is a constant.

We know, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^2 e^{-st}(0)\,dt + \int_2^\infty e^{-st} k\,dt$$

$$= k\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{k}{-s}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{k}{s} e^{-2s}$$

5. Find $L\{e^{-2t}(\cos 3t-\sin 3t)\}$

Given: $L\{e^{-2t}(\cos 3t-\sin 3t)\} = L\{e^{-2t}\cos 3t\} - L\{e^{-2t}\sin 3t\}$

$L\{\cos 3t\} = \frac{s}{s^2+9}$, $L\{\sin 3t\} = \frac{3}{s^2+9}$

By I-shifting theorem, $L\{e^{at}f(t)\} = \bar{f}(s-a)$

$$L\{e^{-2t}(\cos 3t-\sin 3t)\} = \frac{s+2}{(s+2)^2+9} - \frac{3}{(s+2)^2+9} = \frac{s-1}{(s+2)^2+9}$$

6. Find $L\{t^3e^{-4t}\}$

Given: $L\{t^3e^{-4t}\}$

By I-shifting theorem, $L\{e^{at}f(t)\} = \bar{f}(s-a)$

$L\{f(t)\} = L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4} = \bar{f}(s)$

since, $a=-4$, $\bar{f}(s-a) = \bar{f}(s+4)$

$$= \frac{6}{(s+4)^4}$$

7. Evaluate $L\{e^{-t}(t+2)\}$

Given: $L\{e^{-t}(t+2)\}$

$L\{t+2\} = \frac{1}{s^2} + \frac{2}{s} = \bar{f}(s)$

By I-shifting theorem, $L\{e^{at}f(t)\} = \bar{f}(s-a)$, $a=-1$

$$L\{e^{-t}(t+2)\} = \frac{1}{(s+1)^2} + \frac{2}{(s+1)}$$

8. Evaluate $L\{t^2\cosh at\}$

Given: $L\{t^2\cosh at\}$

$L\{\cosh at\} = \frac{s}{s^2-a^2} = \bar{f}(s)$

By multiplication property, $L\{t^nf(t)\} = (-1)^n\frac{d^n}{ds^n}\bar{f}(s)$

$$L\{t^2\cosh at\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2-a^2}\right)$$

$$= \frac{d}{ds}\left(\frac{s^2-a^2-2s^2}{(s^2-a^2)^2}\right) = \frac{d}{ds}\left(\frac{-a^2-s^2}{(s^2-a^2)^2}\right)$$

$$= \frac{-2s(s^2-a^2)^2-(-a^2-s^2)\,2(s^2-a^2)(2s)}{(s^2-a^2)^4}$$

$$= \frac{2s(s^2-a^2)[a^2-s^2+2a^2+2s^2]}{(s^2-a^2)^4}$$

$$= \frac{(s^2-a^2)}{(s^2-a^2)^4}(6a^2s+2s^3)$$

$$= \frac{6a^2s+2s^3}{(s^2-a^2)^3}.$$

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9. Find $L\{\frac{\sin t}{t}\}$

Given: $L\{\frac{\sin t}{t}\}$

$$L\{\sin t\} = \frac{1}{1+s^2} = \bar{f}(s)$$

By division Property, $L\{\frac{f(t)}{t}\} = \int_s^\infty \bar{f}(s)\,ds$

$$L\{\frac{\sin t}{t}\} = \int_s^\infty \frac{1}{1+s^2}\,ds = \tan^{-1} s\Big]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \pi/2 - \tan^{-1}s.$$

10. State necessary & sufficient conditions for existence of Laplace transform.

A function $f(t)$ is said to have its Laplace transform if-

i) $f(t)$ is piece-wise continuous

ii) $f(t)$ is of exponential order

i.e, $\exists$ two constants M & a $\ni |f(t)| \le Me^{at}$ $\forall$ $t \to \infty$

11. Find $L^{-1}\{\frac{1}{s(s^2+9)}\}$

Given: $L^{-1}\{\frac{1}{s(s^2+9)}\} = L^{-1}\{\frac{A}{s} + \frac{Bs+C}{s^2+9}\}$ —①

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$1 = A(s^2+9) + (Bs+C)s$

$1 = As^2 + Bs^2 + Cs + 9A$ — (2)

From (2), $A+B=0$, $A=\frac{1}{9}$, $C=0$.

$B=-A$

$B=-\frac{1}{9}$.

Sub in (1), $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = L^{-1}\left\{\frac{1}{9s}\right\} - L^{-1}\left\{\frac{s}{9(s^2+3^2)}\right\}$

$= \frac{1}{9} - \frac{1}{9}\cos 3t$

$= \frac{1}{9}(1-\cos 3t)$

(Or)

$L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$

$L^{-1}\{\bar{f}(s)\} = 1 = f(u)$ & $L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+3^2}\right\} = \frac{1}{3}\sin 3t$

$= \frac{1}{3}\sin 3(t-u) = g(t-u)$

By Convolution theorem, $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \int_0^t f(u)g(t-u)\,du$

$= \int_0^t \frac{1}{3}\sin 3(t-u)\,du$

$= \frac{1}{9}\cos 3(t-u)\Big]_0^t$

$= \frac{1}{9}[1-\cos 3t]$.

12. Evaluate $L^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\}$

Given: $L^{-1}\left\{\frac{(\sqrt{s}-1)^2}{s^2}\right\} = L^{-1}\left\{\frac{s}{s^2} + \frac{1}{s^2} - \frac{2\sqrt{s}}{s^2}\right\}$

$= L^{-1}\left\{\frac{1}{s} + \frac{1}{s^2} - \frac{2}{s^{3/2}}\right\} = 1 + t - 2L^{-1}\left\{\frac{1}{s^{1+1/2}}\right\}$

$= 1 + t - \frac{2\sqrt{t}}{\frac{1}{2}!}$

$= 1 + t - 4\frac{\sqrt{t}}{\sqrt{\pi}}$

$\left(\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!} \text{ \& } \frac{1}{2}! = \frac{\sqrt{\pi}}{2}\right)$

13. Find $L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$

Given $L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$

$= L^{-1}\left\{\frac{A}{s+2} + \frac{B}{s+3}\right\}$

For A, $s+2=0 \Rightarrow s=-2$

$A = \frac{1}{-2+3} = 1$

For B, $s+3=0 \Rightarrow s=-3$

$B = \frac{1}{-3+2} = -1$

$= L^{-1}\left\{\frac{1}{s+2}\right\} - L^{-1}\left\{\frac{1}{s+3}\right\}$

$= e^{-2t} - e^{-3t}.$

---

14. Evaluate $L^{-1}\left\{\frac{s}{s^2-4}\right\}$

$L^{-1}\left\{\frac{s}{s^2-4}\right\} = L^{-1}\left\{\frac{s}{s^2-2^2}\right\} = \cosh 2t.$ $\quad \boxed{\because L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at}$

---

15. Evaluate $L^{-1}\left\{\frac{5s+10}{9s^2-16}\right\}$

Given: $L^{-1}\left\{\frac{5s+10}{9s^2-16}\right\} = L^{-1}\left\{\frac{5s}{9s^2-16}\right\} + 10L^{-1}\left\{\frac{1}{9s^2-16}\right\}$

$= \frac{5}{9} L^{-1}\left\{\frac{s}{s^2-\left(\frac{4}{3}\right)^2}\right\} + \frac{5}{6} L^{-1}\left\{\frac{4/3}{s^2-\left(\frac{4}{3}\right)^2}\right\}$

We know, $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$ &

$L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

$L^{-1}\left\{\frac{5s+10}{9s^2-16}\right\} = \frac{5}{9}\cosh\frac{4}{3}t + \frac{5}{6}\sinh\frac{4}{3}t.$

---

16. Find $L^{-1}\left\{\frac{3s+2}{(s+1)^3}\right\}$

Given: $L^{-1}\left\{\frac{3s+2}{(s+1)^3}\right\} = L^{-1}\left\{\frac{3(s+1)-1}{(s+1)^3}\right\}$

By Inverse I-shifting theorem, $L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t)$

$a=-1$, $L^{-1}\left\{\frac{3s+2}{(s+1)^3}\right\} = e^{-t} L^{-1}\left\{\frac{3s-1}{s^3}\right\}$

$= e^{-t} L^{-1}\left\{\frac{3}{s^2} - \frac{1}{s^3}\right\}$

$= e^{-t}\left[3t - \frac{t^2}{2}\right]$

---

17. Find $L^{-1}\left\{\frac{e^{-2s}}{s^2-4}\right\}$

Let $f(s) = \frac{1}{s^2-4} = \frac{1}{s^2-2^2}$

$f(t) = L^{-1}\{\bar{f}(s)\} = \frac{\sinh 2t}{2}$, $f(t-a) = f(t-2) = \frac{\sinh 2(t-2)}{2}$

By Inverse II-shifting theorem, $L^{-1}\{e^{-as}\bar{f}(s)\} = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

$L^{-1}\left\{\frac{e^{-2s}}{s^2-4}\right\} = \begin{cases} \frac{\sinh 2(t-2)}{2}, & t>2 \\ 0, & t<2. \end{cases}$

18. Find $L^{-1}\left\{\frac{s+6}{s^2+6s+13}\right\}$

Given: $L^{-1}\left\{\frac{s+6}{s^2+6s+13}\right\} = L^{-1}\left\{\frac{(s+3)+3}{(s+3)^2+4}\right\}$

By Inverse I-shifting theorem, $L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t)$

$$L^{-1}\left\{\frac{(s+3)+3}{(s+3)^2+4}\right\} = e^{-3t}L^{-1}\left\{\frac{s+3}{s^2+4}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{s}{s^2+4} + \frac{3}{s^2+4}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{s}{s^2+2^2} + \frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\}$$

$$= e^{-3t}\left(\cos 2t + \frac{3}{2}\sin 2t\right)$$

19. Solve the initial value problem using Laplace transform $y''+4y=0$, $y(0)=1$, $y'(0)=6$.

Given: $y''+4y=0$, $y(0)=1$ & $y'(0)=6$

Take Laplace transform on both sides,

$$L\{y''(t)\}+4L\{y(t)\}=0$$

Use Laplace transform of derivatives,

$$s^2L\{y(t)\}-sy(0)-y'(0)+4L\{y(t)\}=0.$$

$$s^2L\{y(t)\}-s-6+4L\{y(t)\}=0$$

$$L\{y(t)\}(s^2+4) = s+6.$$

$$L\{y(t)\} = \frac{s+6}{s^2+2^2}$$

$$y(t) = L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{6}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$\therefore y(t) = \cos 2t + 3\sin 2t.$$

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LAQs:

1. Evaluate $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

Given: $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

$L\{\cos at\} = \frac{s}{s^2+a^2}$, $L\{\cos bt\} = \frac{s}{s^2+b^2}$

$L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = \bar{f}(s)$

By division property, $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \frac{s\,ds}{s^2+a^2} - \frac{1}{2}\int_s^\infty \frac{s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\log|s^2+a^2|\Big]_s^\infty - \frac{1}{2}\log|s^2+b^2|\Big]_s^\infty$$

$$= \left[\log\sqrt{s^2+a^2} - \log\sqrt{s^2+b^2}\right]_s^\infty$$

$$= \log\sqrt{\frac{s^2+a^2}{s^2+b^2}}\Bigg]_s^\infty$$

$$= \log 1 - \log\sqrt{\frac{s^2+a^2}{s^2+b^2}}$$

$$= \log\sqrt{\frac{s^2+a^2}{s^2+b^2}}$$

2. Evaluate $L\left\{\int_0^t \frac{\sin 3u}{u}\,du\right\}$

$L\{\sin 3t\} = \frac{3}{s^2+9} =$

By division property, $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$

$L\left\{\frac{\sin 3t}{t}\right\} = \int_s^\infty \frac{3}{s^2+3^2}\,ds$

$$= 3\int_s^\infty \frac{1}{s^2+3^2}$$

$$= \frac{3}{3}\,Tan^{-1}\left(\frac{s}{3}\right)\Big]_s^\infty$$

$$= \frac{3}{3}\left(Tan^{-1}\infty - Tan^{-1}\frac{s}{3}\right)$$

$$= \frac{\pi}{2} - Tan^{-1}\frac{s}{3} \quad = \bar{f}(s)$$

Now, $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$

$$L\left\{\int_0^t \frac{\sin 3u}{u}\,du\right\} = \frac{1}{s}\left[\frac{\pi}{2} - Tan^{-1}\frac{s}{3}\right]$$

---

3. Evaluate $L\left\{e^{-t}\int_0^t \frac{\sin u}{u}\,du\right\}$

$L\{\sin t\} = \frac{1}{s^2+1}$

By Division property, $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds = Tan^{-1}s\Big]_s^\infty$

$$= \frac{\pi}{2} - Tan^{-1}s$$

Now, $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$

$$L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\left(\frac{\pi}{2} - Tan^{-1}s\right)$$

By I-shifting theorem, $L\{e^{at}f(t)\} = \bar{f}(s-a)$

$$L\left\{e^{t}\int_0^t \frac{\sin u}{u}\,dt\right\} = \frac{1}{s+1}\left(\frac{\pi}{2} - Tan^{-1}(s+1)\right)$$

---

4. Evaluate $L\left\{\int_0^t e^u \frac{\sin u}{u}\,du\right\}$

$L\{\sin t\} = \frac{1}{1+s^2}$

$L\{e^t \sin t\} = \frac{1}{1+(s-1)^2}$ (By FST).

By division property, $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \frac{1}{1+(s-1)^2}\,ds = Tan^{-1}(s-1)\Big]_s^\infty$

$= \frac{\pi}{2} - Tan^{-1}(s+1)$

Now, $L\{\int_0^t f(u)\,dt\} = \frac{1}{s}\bar{f}(s)$

$= \frac{1}{s}\left(\frac{\pi}{2} - Tan^{-1}(s+1)\right)$

5. Evaluate $L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\}$

$L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\} = L^{-1}\left\{\frac{A}{s-3} + \frac{Bs+C}{s^2+4}\right\}$

$A(s^2+4)+(Bs+C)(s-3) = s \Rightarrow As^2+4A+Bs^2-3Bs+Cs-3s = s$

$A+B=0, \quad 4A-3C=0, \quad -3B+C=1$

By solving above equations we get,

$A = \frac{3}{13}, \quad B = -\frac{3}{13}, \quad C = \frac{4}{13}$

$L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\} = L^{-1}\left\{\frac{3}{13(s-3)}\right\} + L^{-1}\left\{\frac{-3s+4}{13(s^2+4)}\right\}$

$= \frac{3}{13}e^{3t} + L^{-1}\left\{\frac{-3s}{13(s^2+2^2)}\right\} + \frac{4}{13}L^{-1}\left\{\frac{2}{2(s^2+2^2)}\right\}$

$= \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t.$

6. Find $L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{5(s+1)-2}{((s+1)-2)((s+1)^2+4)}\right\}$

By FST, $= e^{-t}L^{-1}\left\{\frac{5s-2}{(s-2)(s^2+4)}\right\}$

$= e^{-t}L^{-1}\left\{\frac{A}{s-2} + \frac{Bs+C}{s^2+4}\right\}$

$\frac{A}{s-2} + \frac{Bs+C}{s^2+4} = \frac{A(s^2+4)+(Bs+C)(s-2)}{(s-2)(s^2+4)} = \frac{5s-2}{(s-2)(s^2+4)}$

$As^2+4A+Bs^2-2Bs+Cs-2C = 5s-2$

On comparing both sides, we get $A+B=0$, $4A-2C=-2$ —(1), $-2B+C=5$ —(2)

$A=-B$

Solving (1) & (2), $4A-2C=-2$

$4A+2C=10$ (Multiply (2) by '2')

$A = 8/8 = 1$

$A=1, B=-1, C=3$

$= e^{-t}L^{-1}\left\{\frac{1}{s-2} + \frac{(-s+3)}{s^2+2^2}\right\} = e^{-t}\left[e^{2t} - \cos 2t + \frac{3}{2}\sin 2t\right]$

7. Evaluate $L^{-1}\left\{\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}$

Let $\bar{f}(s) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

$= \log(s^2+b^2) - \log(s^2+a^2)$

Differentiate w.r.t s,

$$\bar{f}'(s) = \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$$

Take L^{-1} on both sides

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right\} = 2\cos bt - 2\cos at$$

By Inverse Laplace transform of derivatives,

$$L^{-1}\{\bar{f}'(s)\} = -tf(t)$$

$$2\cos bt - 2\cos at = -tf(t)$$

$$f(t) = \frac{2(\cos at - \cos bt)}{t}$$

---

8. Evaluate $L^{-1}\left\{\log\left(\frac{s+3}{s+4}\right)\right\}$

Let $\bar{f}(s) = \log\left(\frac{s+3}{s+4}\right) = \log(s+3) - \log(s+4)$

Differentiate w.r.t s,

$$\bar{f}'(s) = \frac{1}{s+3} - \frac{1}{s+4}$$

take L^{-1} on both sides.

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{1}{s+3} - \frac{1}{s+4}\right\} = e^{-3t} - e^{-4t}$$

By I.L.T of derivatives, $L^{-1}\{\bar{f}'(s)\} = -tf(t)$

$$L^{-1}\{\bar{f}'(s)\} = -tf(t)$$

$$e^{-3t} - e^{-4t} = -tf(t)$$

$$f(t) = \frac{e^{-4t} - e^{-3t}}{t}$$

9. Find $L^{-1}\left\{\frac{e^{-as}}{s^2+1}+\frac{e^{-s}}{s^3}\right\}$

$$L^{-1}\left\{\frac{e^{-as}}{s^2+1}+\frac{e^{-s}}{s^3}\right\} = L^{-1}\left\{\frac{e^{-as}}{s^2+1}\right\}+L^{-1}\left\{\frac{e^{-s}}{s^3}\right\}$$

Compare with inverse II-shift theorem

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$$

$$\therefore g(t) = \begin{cases} \sin(t-a), & t>a \\ 0, & t<a \end{cases}$$

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$$L^{-1}\left\{\frac{e^{-s}}{s^3}\right\}.$$

$$\bar{f}(s) = \frac{1}{s^3}$$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2!}$$

$$f(t) = \frac{t^2}{2}$$

$$\therefore g(t) = \begin{cases} \frac{(t-1)^2}{2}, & t>1 \\ 0, & t<1. \end{cases}$$

---

10. Evaluate $L^{-1}\left\{\cot^{-1}\left(\frac{s+3}{2}\right)\right\}$

By Inverse F.S.T,

$$L^{-1}\left\{\cot^{-1}\left(\frac{s+3}{2}\right)\right\} = e^{-3t} L^{-1}\left\{\cot^{-1}\left(\frac{s}{2}\right)\right\}$$

We know, by multiplication property if $L\{f(t)\}=\bar{f}(s)$

then $L\{t.f(t)\} = -\frac{d}{ds}(\bar{f}(s))$

$$f(t) = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\}.$$

$$L^{-1}\left\{\cot^{-1}\left(\frac{s+3}{2}\right)\right\} = \frac{e^{-3t}}{-t} L^{-1}\left\{\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\}$$

$\because \frac{d}{dx}\cot^{-1}x = \frac{1}{1+x^2}$

$$= -\frac{e^{-3t}}{t}L^{-1}\left\{\frac{-1}{1+(s/2)^2}\cdot\frac{1}{2}\right\}$$

$$= \frac{-e^{3t}}{t}L^{-1}\left\{\frac{-2}{2^2+s^2}\right\}$$

$$= \frac{e^{-3t}}{t}\sin 2t.$$

11. Apply convolution theorem to find $L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$

$$L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{(s-1)}\cdot\frac{1}{(s+2)}\right\}$$

$$L^{-1}\left\{\frac{1}{s-1}\right\} = e^{t} = f(t) \quad \& \quad L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t} = e^{-2(t-u)} = g(t-u)$$

From convolution theorem,

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t e^{u}\cdot e^{-2(t-u)}\,du$$

$$= \int_0^t e^{u-2t+2u}\,du$$

$$= \int_0^t e^{3u-2t}\,du$$

$$= \left.\frac{e^{3u-2t}}{-2}\right]_0^t$$

$$= \frac{e^{t}}{-2} - \frac{e^{-2t}}{-2}$$

$$= \frac{1}{2}\left[e^{-2t} - e^{t}\right]$$

12. Find the inverse laplace transform of $\bar{f}(s) = \frac{s}{(s^2+a^2)^2}$ by using Convolution theorem.

Given: $\bar{f}(s) = \frac{s}{(s^2+a^2)^2}$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{(s^2+a^2)}\cdot\frac{s}{(s^2+a^2)}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at = f(t), \quad L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = \cos a(t-u) = g(t-u)$$

$$\frac{1}{a}\sin au = f(u)$$

By CT, $L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du$

$$= \int_0^t \frac{1}{a}\sin au\,\cos a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t 2\sin au\cos a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t \sin(au+at-au) + \sin(au-at+au)\,du$$

$$= \frac{1}{2a}\int_0^t (\sin at + \sin(2au-at))\,du$$

$$= \frac{1}{2a}\left[u\sin at + \frac{(-\cos(2au-at))}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\cos at - 0 + \frac{1}{2a}\cos(-at)\right]$$

$$= \frac{1}{2a}\left[t\sin at - \cancel{\frac{1}{2a}\cos at} + \cancel{\frac{1}{2a}\cos at}\right]$$

$$= \frac{1}{2a} t\sin at.$$

13. Solve $y''+3y'+2y=3$, $y(0)=1$, $y'(0)=1$ by applying Laplace transform.

Given: $y''+3y'+2y=3$.

Apply L.T on b.s, $L\{y''\}+3L\{y'\}+2L\{y\}=3L\{1\}$.

$$\Rightarrow s^2L\{y\}-s\,y(0)-y'(0)+3[sL\{y\}-y(0)]+2L\{y\}=\frac{3}{s}$$

$$\Rightarrow s^2L\{y\}-s-1+3sL\{y\}-3+2L\{y\}=\frac{3}{s}$$

$$\Rightarrow L\{y\}[s^2+3s+2]-s-4=\frac{3}{s}$$

$$\Rightarrow L\{y\}[s^2+3s+2]=\frac{3}{s}+s+4=\frac{3+s^2+4s}{s}$$

$$\Rightarrow L\{y\}=\frac{s^2+4s+3}{s(s^2+3s+2)}.$$

$$\Rightarrow y(t)=L^{-1}\left\{\frac{s^2+4s+3}{(s^2+3s+2)s}\right\}=L^{-1}\left\{\frac{s^2+4s+3}{s(s+1)(s+2)}\right\}$$

$$=L^{-1}\left\{\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}\right\}$$

For A, $s=0$ then $A=\frac{3}{(0+1)(0+2)}=\frac{3}{2}$.

For B, $s=-1$ then $B=0$

For C, $s=-2$ then $C=\frac{4-8+3}{(-2)(-2+1)}=\frac{-1}{2}$

$$=L^{-1}\left\{\frac{3}{2s}\right\}+0-\frac{1}{2}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$=\frac{3}{2}(1)+0-\frac{1}{2}e^{-2t}$$

$$=\frac{3}{2}-\frac{1}{2}e^{-2t}$$

$$=\frac{1}{2}\left[3-e^{-2t}\right]$$

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14. Solve $y''+y=2e^t$, $y(0)=0$, $y'(0)=2$, by using L.T.

Given: $y''+y=2e^t$

Apply L.T on both sides

$$L\{y''(t)\}+L\{y(t)\}=2L\{e^t\}$$

$$s^2L\{y(t)\}-sy(0)-y'(0)+L\{y(t)\}=2\left(\frac{1}{s-1}\right)$$

$$L\{y(t)\}[s^2+1]-s(0)-2=\frac{2}{s-1}$$

$$L\{y(t)\}(s^2+1)=\frac{2}{s-1}+2=\frac{2+2s-2}{s-1}$$

$$L\{y(t)\}=\frac{2s}{(s-1)(s^2+1)}$$

$$y(t)=L^{-1}\left\{\frac{2s}{(s-1)(s^2+1)}\right\}.$$

$$y(t)=2L^{-1}\left\{\frac{1}{(s-1)}\cdot\frac{s}{s^2+1}\right\}$$

$$L^{-1}\left\{\frac{1}{s-1}\right\}=e^t=e^u=f(u)$$

$$L^{-1}\left\{\frac{s}{s^2+1}\right\}=\cos t=\cos(t-u)=g(t-u)$$

By C.T, $L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}=\int_0^t f(u)g(t-u)\,du$

Let $A=\int_0^t e^u\cos(t-u)\,du$

$$=\cos(t-u)\int_0^t e^u\,du-\int_0^t\left(\frac{d}{dx}\cos(t-u)\int_0^t e^u\,du\right)du$$

$$=\cos(t-u)e^u\Big]_0^t-\int_0^t\sin(t-u)e^u\,du$$

$$=\cos(t-u)e^u\Big]_0^t-\left[\sin(t-u)e^u\Big]_0^t+\int_0^t\cos(t-u)e^u\,du\right]$$

$$=\frac{\pi}{2}e^t-\cos t-(0-\sin t)-\int_0^t\cos(t-u)e^u\,du$$

$$=\frac{\pi}{2}e^t-\cos t+\sin t-A$$

$$2A=\frac{\pi}{2}e^t-\cos t+\sin t$$

$$A=\frac{\pi}{4}e^t+\frac{1}{2}(\sin t-\cos t)$$

15. Solve $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t$ where $y(0)=2$, $y'(0)=-1$ by method of L.T.

Given: $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t$, $y(0)=2$ & $y'(0)=-1$

Apply L.T on both sides,

$$L\{y''(t)\} - 2L\{y'(t)\} + L\{y(t)\} = L\{e^t\}$$

$$s^2L\{y(t)\} - s\,y(0) - y'(0) - 2[s.L\{y(t)\} - y(0)] + L\{y(t)\} = \frac{1}{s-1}$$

$$L\{y(t)\}(s^2-2s+1) - 2s - (-1) + 4 = \frac{1}{s-1}$$

$$L\{y(t)\}(s^2-2s+1) - 2s + 5 = \frac{1}{s-1}$$

$$L\{y(t)\}(s^2-2s+1) = \frac{1}{s-1} + 2s - 5 = \frac{2s^2-7s+6}{s-1}$$

$$L\{y(t)\} = \frac{2s^2-7s+6}{(s-1)(s^2-2s+1)}$$

$$L\{y(t)\} = \frac{2s^2-7s+6}{(s-1)^3}$$

$$y(t) = L^{-1}\left\{\frac{2s^2-7s+6}{(s-1)^3}\right\}$$

$$\frac{2s^2-7s+6}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$$2s^2-7s+6 = A(s-1)^2 + B(s-1) + C$$

$$2s^2-7s+6 = A(s^2-2s+1) + B(s-1) + C$$

Coeff of $s^2 \Rightarrow A = 2$

Coeff of $s \Rightarrow -2A + B = -7 \Rightarrow B = -7+4 = -3$

Constants $\Rightarrow A - B + C = 6 \Rightarrow C = 6-3-2 = 1$

$$y(t) = L^{-1}\left\{\frac{2}{s-1}\right\} - 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= 2e^t - 3e^tL^{-1}\left\{\frac{1}{s^2}\right\} + e^tL^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 2e^t - 3e^tt + e^t\frac{t^2}{2}$$

$$= e^t\left[2 - 3t + \frac{t^2}{2}\right]$$

$$= \frac{e^t}{2}[t^2 - 6t + 4]$$

16. Solve $\frac{d^2y}{dt^2}+9y=\cos 2t$ if $y(0)=1, y(\frac{\pi}{2})=-1$ by the method of L.T.

Since $y'(0)$ is not given, let $y'(0)=A$

$$y''(t)+9y(t)=\cos 2t$$

Apply L.T on b.s,

$$L\{y''(t)\}+9L\{y(t)\}=L\{\cos 2t\}$$

$$s^2L\{y(t)\}-sy(0)-y'(0)+9L\{y(t)\}=L\{\cos 2t\}$$

$$L\{y(t)\}[s^2+9]-s-A=\frac{s}{s^2+4}$$

$$L\{y(t)\}=\left[\frac{s}{s^2+4}+s+A\right]\left[\frac{1}{s^2+9}\right]$$

$$=\frac{s}{(s^2+4)(s^2+9)}+\frac{s}{(s^2+9)}+\frac{A}{(s^2+9)}$$

Now partial Fraction, $\frac{s}{(s^2+4)(s^2+9)}=\frac{As+B}{s^2+9}+\frac{Cs+D}{s^2+4}$

$$s=(As+B)(s^2+4)+(Cs+D)(s^2+9)$$

$$s=As^3+Bs^2+4As+4B+Cs^3+Ds^2+9Cs+9D$$

Compare coeff of $s^3 \Rightarrow A+C=0$ — (1)

Coeff of $s^2 \Rightarrow B+D=0$ — (2)

Coeff of $s \Rightarrow 4A+9C=1$ — (3)

constants $\Rightarrow 4B+9D=0$ — (4)

Now, $4\times(1)-(3) \Rightarrow -5C=-1 \Rightarrow C=1/5$

From (1) $\Rightarrow A=-1/5$

From (2) & (4) $\Rightarrow B=0$ & $D=0$

$$L\{y(t)\}=\frac{A}{s^2+9}+\frac{s}{s^2+9}-\frac{1}{5}\frac{s}{s^2+9}+\frac{1}{5}\frac{s}{s^2+4}$$

$$=\frac{A}{s^2+9}+\frac{4}{5}\cdot\frac{s}{s^2+9}+\frac{1}{5}\frac{s}{s^2+4}$$

$$y(t)=AL^{-1}\left\{\frac{1}{s^2+9}\right\}+\frac{4}{5}L^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{1}{5}L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$=\frac{A}{3}\sin 3t+\frac{4}{5}\cos 3t+\frac{1}{5}\cos 2t$$

Given, $y(\pi/2)=-1$

$$-1=\frac{A}{3}\sin 3\pi/2+\frac{4}{5}\cos 3\pi/2+\frac{1}{5}\cos 2\pi/2$$

$$-1=-\frac{A}{3}+0-\frac{1}{5} \Rightarrow \frac{A}{3}=1-\frac{1}{5}=\frac{4}{5} \Rightarrow \frac{A}{3}=\frac{4}{5}.$$

$$\therefore\ y(t)=\frac{4}{5}\sin 3t+\frac{4}{5}\cos 3t+\frac{1}{5}\cos 2t.$$