Probability and Statistics

# Introduction to probability
## (UNIT-1)

→ Permutation:- If r items/elements are selected from n elements and order effects, it is called permutation. It is denoted by $^{n}p_{r}$

→ Combination:- If r elements are selected from n elements and order doesn't effect, it is called combination. It is denoted by $^{n}C_{r}$

→ ① Random experiment:- If an experiment is conducted repeatedly & outcome of that exp is not same every time is called random experiment.

→ ② Outcome of an experiment:-
Result of a random exp. is called outcome.

③ Event:-
An outcome or combination of outcomes of a random experiment is called event.

④ Disjoint (or) Mutually exclusive event:-
If happening of one event doesn't effect happeneing of other event is called disjoint event.

## (5) Sample space:-

The collection of all outcomes of a random experiment is called sample space.

→ Definition of probability of an event

$$\text{Probability}(E) = \frac{\text{Favourable events}}{\text{Total number of events}}$$

**Eg:-1** Toss a die

– Example of a random exp. is tossing of die

$S = \{1, 2, 3, 4, 5, 6\}$ $\quad$ $S \rightarrow$ Sample Space

$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$ $\quad$ $P(\text{prime}) = \frac{3}{6} = \frac{1}{2}$

$P(>2) = \frac{4}{6} = \frac{2}{3}$

**Eg:-2** Tossing 4 coins

S = { HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THTH, TTHH, THHT, HTTT, THTT, TTHT, TTTH, TTTT }

$P(2\text{ heads}) = \frac{6}{16} = \frac{3}{8}$ $\quad$ $P(>2H) = \frac{5}{16}$

$$P(2\text{ tails}) = \frac{6}{16} = \frac{3}{8}$$

Eg:-3 Tossing two dice

$S = \{36\}$

(1,1)
(1,2),(2,1)
(1,3)(2,2)(3,1)
(1,4)(2,3)(3,2)(4,1)
(1,5)(2,4)(3,3)(4,2)(5,1)
(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)
(2,6)(3,5)(4,4)(5,3)(6,2)
(3,6)(4,5)(5,4)(6,3)
(4,6)(5,5)(6,4)
(5,6)(6,5)
(6,6)

$$P(\text{Sum of numbers is } 8) = \frac{5}{36}$$

$$P(\text{first} > \text{second}) = \frac{15}{36}$$

$$P(\text{both no!s are same}) = \frac{6}{36} = \frac{1}{6}$$

★ Properties of probability:-

1.) Probability of an event lies between 0 and 1

$$0 < P(E) < 1$$

2.) If $P(E) = 0$, it is an impossible event.

3.) If $P(E) = 1$, it is a certain (or) definite event.

4.) Probability of sample space, $P(S.S) = 1$

5.) $P(A) = 1 - P(\bar{A})$

$$P(A) + P(\bar{A}) = 1$$

6.) If A and B are disjoint events, then

$$P(A \cup B) = P(A) + P(B)$$

7.) If A and B are any two events then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is also called Addition law of probability.

8.) If A and B are two independent events then

$$P(A \cap B) = P(A) . P(B)$$

* Conditional probability:-

Let A and B are any two events. Then

$$P(B/A) = \frac{P(A\cap B)}{P(A)}$$

where $P(B/A)$ is called Conditional probability. It defines as probability of happening of event B on the condition that event A already occured.

Note:- If A and B are independent events then $P(B/A) = P(B)$

Similarly, $P(A/B) = \frac{P(A\cap B)}{P(B)}$

If A and B are independent events then $P(A/B) = P(A)$

Q) A pair of fair dice are thrown. Find the prob. that sum of the numbers is 10 or greater if 5 appears on first die.

Sol:- (5,1) (5,2) (5,3) (5,4) (5,5) (5,6)

$\therefore P(E) = \frac{2}{6} = \frac{1}{3}$

Q) A bag contains 5 white and 3 black balls. Two balls are drawn one after another without replacement. Find the prob that second ball drawn is black given that first ball drawn is white

Sol:- $P(E) = 3/7$

5W
3B

Q) From a city population, the prob of selecting
i) A male or a smoker is $7/10$
ii) A male smoker is $2/5$ and
iii) A male, if a smoker is already selected is $2/3$
Find the probability of selecting a) A non-smoker
b) A male c) a smoker if a male is already selecting

Sol:- A = A male B = A smoker
i) $P(A \cup B) = 7/10$, ii) $P(A \cap B) = 2/5$, iii) $P(A/B) = 2/3$

a) $P(\overline{B}) = 1 - P(B)$

$\left[\because P(A/B) = \frac{P(A \cap B)}{P(B)}\right]$

$= 1 - \frac{P(A \cap B)}{P(A/B)} = 1 - \frac{2/5}{2/3}$

$= 1 - \frac{3}{5} = \frac{2}{5}$

b) & c) on next page

Q) A natural number is selected at random from first 100 natural numbers. What is the probability that, ~~the~~ the second digit of number so selected is greater than 6, given that first digit is four.

Sol- Event A - a 2-digit number first digit is 4

Event B - a " " " whose second digit is greater than 6.

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⋮
49 (10)

$P(B/A) = 3/10$

b) $P(A) = ?$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\frac{7}{10} = P(A) + \frac{3}{5} - \frac{2}{5} \Rightarrow P(A) = \frac{7}{10} - \frac{1}{5}$

$= \frac{5}{10} = \frac{1}{2}$

c) $P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{2/5}{1/2}$

$= \frac{4}{5}$

Q) 60% of employees of XYZ corporation are college graduates. Of these 10% are in Sales. Of the employees who didn't graduate from college, 80% are in sales. What is the prob. that

i) an employee selected at random is in sales

ii) an " " " " is neither in sales nor a college graduate

Sol:- A = An Employee is a college graduate   B = An Employee is in Sales.

$P(A) = \frac{60}{100} = 0.6$

$P(B/A) = \frac{10}{100} = 0.1$

$P(B/\bar{A}) = 0.8$

i) $P(B) = P(A \cap B) + P(\bar{A} \cap B)$

$= P(A) \cdot P(B/A) + P(\bar{A}) \cdot P(B/\bar{A})$

$\left[ \because P(B/A) = \frac{P(A \cap B)}{P(A)} \right]$

$P(B/\bar{A}) = \frac{P(\bar{A} \cap B)}{P(\bar{A})}$

$= (0.6 \times 0.1) + (0.4)(0.8)$

$= 0.06 + 0.32$

$= 0.38$

ii) $P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [0.6 + 0.38 - 0.06]$

$= 0.08$

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* Baye's Theorem

If $E_1, E_2, \ldots E_n$ are mutually disjoint events with $P(E_i) \neq 0$ for each $i$, then for any arbitrary event $A$ which is a subset of $\bigcup_{i=1}^{n} E_i$ i.e $A \subset \bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have then

$$P(E_i/A) = \frac{P(E_i) \times P(A/E_i)}{P(A)} = \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(E_i).P(A/E_i)}$$

(or)

→ Let $E_1, E_2, \ldots E_n$ are mutually disjoint events, $P(E_i) \neq 0$ for $i = 1$ to $n$ and $A$ be an event such that $A \subset \bigcup_{i=1}^{n} E_i$ & $P(A) > 0$ then

$$P(E_i/A) = \frac{P(E_i).P(A/E_i)}{P(A)} = \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(E_i).P(A/E_i)}$$

Proof:- Given that

$$A \subset \bigcup_{i=1}^{n} E_i \quad \text{then } A = A \cap \left(\bigcup_{i=1}^{n} E_i\right)$$

$$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \ldots \cup (A \cap E_n)$$

$$A = \bigcup_{i=1}^{n} (A \cap E_i)$$

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i)\right]$$

Here each event is mutually disjoint

$\therefore \quad P(A) = \sum_{i=1}^{n} P(A \cap E_i)$

$$P(A) = \sum_{i=1}^{n} P(A/E_i).P(E_i)$$

$\left[\because P(A/B) = \frac{P(A \cap B)}{P(B)}\right.$

$\left.P(A/E_i) = \frac{P(A \cap E_i)}{P(E_i)}\right]$

Now, $P(E_i/A) = \dfrac{P(E_i \cap A)}{P(A)}$

$$= \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(E_i).P(A/E_i)}$$

Hence proved.

1.) The contents of urn's 1, 2, 3 are as follows:

urn 1: One white, two black, three red balls

urn 2: 2 white, 1 Black, and 1 red balls

urn 3: 4 white, 5 Black and 3 red balls

One box is chosen at random & two balls are drawn from it. They happened to be white and red. What is the prob. that they come from box 1 (or) urn 1

Sol:- $E_1$ = Selection of urn 1

$E_2$ = Selection of urn 2

$E_3$ = Selection of urn 3

A = Two balls are drawn in which one is white and one is red.

$P(E_1) = P(E_2) = P(E_3) = 1/3$

$$P(A/E_1) = \frac{1C_1 \times 3C_1}{6C_2} = \frac{3}{15} = \frac{1}{5}$$

$$P(A/E_2) = \frac{2C_1 \times 1C_1}{4C_2} = \frac{2}{6} = \frac{1}{3}$$

$$P(A/E_3) = \frac{4C_1 \times 3C_1}{12C_2} = \frac{12}{66} = \frac{2}{11}$$

Prob of selecting one white & one red ball from urn 3.

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{\sum_{i=1}^{3} P(E_i).P(A/E_i)}$$

$$= \frac{1/3 \times \frac{1}{5}}{\frac{1}{3}.\frac{1}{5} + \frac{1}{3}.\frac{1}{3} + \frac{1}{3}.\frac{2}{11}} =$$

2) A factory produces a certain type of output by three types of machines. The respective daily production are
Machine 1: 3000 units, Machine 2: 2500 units, and Machine 3: 4500 units. Past experience shows that 1% of output produced by machine 1 is defective, the corresponding fraction of defectives from other two machines are 1.2% and 2% respectively. An item is drawn at random from a days production & found to be defective. What is the prob that it comes from output of i) Machine 1 ii) Machine 2 iii) Machine 3

Sol:- $E_1$ = Production from machine 1

$E_2$ = " " " 2

$E_3$ = " " " 3

A = Defective output.

$P(E_1) = \frac{3000}{10000} = \frac{3}{10}$ , $P(E_2) = \frac{2500}{10000} = \frac{1}{4}$

$P(E_3) = \frac{4500}{10000} = \frac{9}{20}$

$P(A/E_1) = \frac{1}{100}$ , $P(A/E_2) = \frac{1.2}{100}$ , $P(A/E_3) = \frac{2}{100}$

i) $P(E_1/A) = \dfrac{P(E_1).\,P(A/E_1)}{\sum_{i=1}^{3} P(E_i)\,P(A/E_i)}$

$= \dfrac{\frac{3}{10}\cdot\frac{1}{100}}{\frac{3}{10}\cdot\frac{1}{100} + \frac{1}{4}\cdot\frac{1.2}{100} + \frac{9}{20}\cdot\frac{2}{100}}$

$= \dfrac{\frac{3}{1000}}{\frac{3}{1000} + \frac{(1.2)\times 25}{10000} + \frac{45}{100}\cdot\frac{2}{100}}$

$= \dfrac{3}{3+\;3+9} = \dfrac{3}{15} = \dfrac{1}{5}$

ii) $P(E_2/A) = \dfrac{P(E_2)\,P(A/E_2)}{P(E_1).\,P(A/E_1) + P(E_2).\,P(A/E_2) + P(E_3).\,P(A/E_3)}$

$= \dfrac{\frac{25}{100}\cdot\frac{1.2}{100}}{\frac{3}{10}\cdot\frac{1}{100} + \frac{1.2}{100}\times\frac{25}{100} + \frac{45}{100}\cdot\frac{2}{100}}$

$= \dfrac{30}{30+30+90} = \dfrac{30}{150} = \dfrac{1}{5}$

3.) A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope just two consecutive letters/alphabets TA are visible. What is the prob. that the letter came from TATANAGAR.

Sol:- $E_1$ = Letter from CALCUTTA

$E_2$ = " " TATANAGAR

A = Two consecutive alphabets TA are visible on the envelope

$P(E_1) = P(E_2) = 1/2$

$P(A/E_1) = \frac{1}{7}$ → total two cons alphabets

$P(A/E_2) = \frac{2}{8} = \frac{1}{4}$

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2).\,P(A/E_2)}{P(E_1).\,P(A/E_1) + P(E_2).\,P(A/E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{4}}{\frac{1}{2}\cdot\frac{1}{7} + \frac{1}{2}\cdot\frac{1}{4}} = \frac{\frac{1}{8}}{\frac{1}{14} + \frac{1}{8}}$$

$$= \frac{\frac{1}{4}}{\frac{1}{7} + \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{11}{28}} = \frac{28}{4\times 11}$$

$$= \frac{7}{11}$$

4) From a vessel containing 3 white and 5 black balls, 4 four balls are transferred into an empty vessel. From this vessel, a ball is drawn & is found to be white. What is the prob. that out of four balls transferred, 3 are white, 1 is black?

Sol:- $E_1$ = Transfer of 0 White and 4 Black balls

$E_2$ = " 1 White " and 3 Black balls

$E_3$ = Transfer of 2W and 2B

$E_4$ = " " 3W and 1B

A = 1 white ball is drawn from II[nd] vessel.

$$P(E_1) = \frac{3C_0 \times 5C_4}{8C_4} = \frac{1}{14}$$

$$P(E_2) = \frac{3C_1 \times 5C_3}{8C_4} = \frac{3}{7}$$

$$P(E_3) = \frac{3C_2 \times 5C_2}{8C_4} = \frac{3}{7} \qquad P(E_4) = \frac{3C_3 \times 5C_1}{8C_4} = \frac{1}{14}$$

$$P(A/E_1) = 0 \qquad P(A/E_2) = \frac{1C_1}{4C_1}$$

$$P(A/E_3) = \frac{2C_1}{4C_1} \qquad P(A/E_4) = \frac{3C_1}{4C_1}$$

$$P(E_4/A) = \frac{P(E_4).\,P(A/E_4)}{\sum_{i=1}^{4} P(E_i).\,P(A/E_i)}$$

$$= \frac{\frac{1}{14}\cdot\frac{3}{4}}{\frac{1}{14}.0 + \frac{3}{7}\cdot\frac{1}{4} + \frac{3}{7}\cdot\frac{2}{4} + \frac{1}{14}\cdot\frac{3}{4}}$$

$$= \frac{1}{7}$$

5.) A man speaks truth 4 out of 5 times. A die is rolled. He reports that there is a six. What is the prob. that it is actually a six?

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Sol: $E_1$ = Man speaks truth

$E_2$ = Man speaks false

A = He reports six on a die

$P(E_1) = \frac{4}{5}$ $P(E_2) = \frac{1}{5}$

$P(A/E_1) = 1$ $P(A/E_2) = \frac{5}{6}$ → In 5 diff. ways, the man can speak lie

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)}$$

$$= \frac{\frac{4}{5}.1}{\frac{4}{5}.1 + \frac{1}{5}.\frac{5}{6}} = \frac{4}{4+\frac{5}{6}}$$

$$= \frac{24}{29}$$

6) Suppose that a product is produced in 3 factories X, Y and Z. It is known that factory X produces thrice as many items as factory Y and factory Y and Z produces same number of item. Assume that 3% of items produced by each of the factories

X and Z are defective while 5% of those manufactured by factory Y are defective. All the items produced in three factories and stocked and an item of product is selected at random

i) What is the prob. that this item is defective

ii) If an item selected at random is found to be defective, what is the prob. that it was produced by a) X, b) Y and c) Z respectively

Sol:- $E_1$ = production from factory X

$E_2$ = " " " Y

$E_3$ = " " " Z

A = Defective output of whole company.

$P(E_1) = \frac{3}{5}$  $P(E_2) = \frac{1}{5}$.  $P(E_3) = \frac{1}{5}$

$P(A/E_1) = \frac{3}{100}$  $P(A/E_2) = \frac{5}{100}$

$P(A/E_3) = \frac{3}{100}$.

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{\sum_{i=1} P(E_i).P(A/E_i)} = \frac{\frac{3}{5}\times\frac{3}{100}}{\frac{3}{5}\times\frac{3}{100}+\frac{1}{5}\times\frac{5}{100}+\frac{1}{5}\times\frac{3}{100}}$$

$$= \frac{9}{9+5+3} = \frac{9}{17}$$

$$P(E_2/A) = \frac{P(E_2).P(A/E_2)}{\sum P(E_i).P(A/E_i)}$$

$$= \frac{\frac{1}{5}\times\frac{5}{100}}{\frac{3}{5}\times\frac{3}{100}+\frac{1}{5}\times\frac{5}{100}+\frac{1}{5}\times\frac{3}{100}}$$

$$= \frac{5}{17}$$

$$P(E_3/A) = \frac{3}{17}$$

7.) There are three boxes having the following compositions of Black and White balls

Box 1: 7W, 3B

Box 2: 4W, 6B

Box 3: 2W, 8B

One of these boxes is chosen at random with probabilities 0.2, 0.6, and 0.2 respectively. From the chosen box, two balls are drawn at random without replacement. Calculate the prob. that both these balls are white.

Sol:- $E_1$ - box 1 $E_2$ - box 2 $E_3$ - box 3

$P(E_1) = 0.2$ $P(E_2) - 0.6$ $P(E_3) = 0.2$

A = 2 white balls

$P(A/E_1) = \frac{7C_2}{10C_2}$ $P(A/E_2) = \frac{4C_2}{10C_2}$ $P(A/E_3) = \frac{2C_2}{10C_2}$

$P(A) = 0.2 \times \frac{7C_2}{10C_2} + 0.6 + \frac{4C_2}{10C_2} + 0.2 \times \frac{2C_2}{10C_2}$

$= 0.2 \times \frac{7}{15} + 0.6 \times \frac{2}{15} + 0.2 \times \frac{1}{45}$

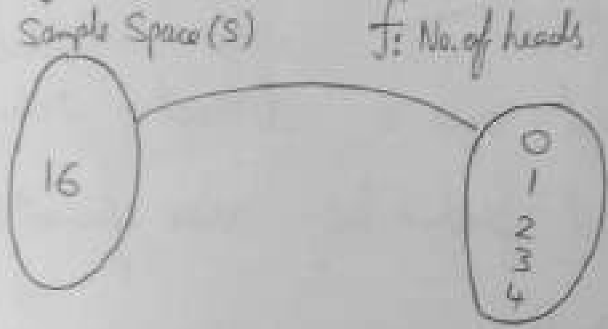
$= \frac{4.2 + 3.6 + 0.2}{45}$

$= \frac{8}{45}$

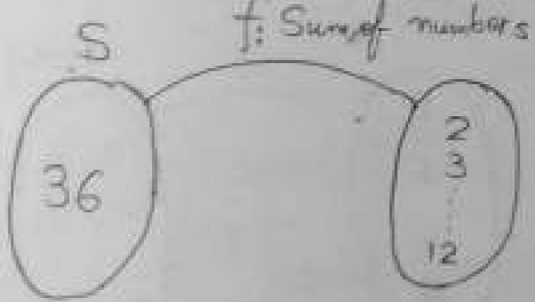
## ★ Random Variable

Random variable is a real valued function defined over a sample space of a random experiment (or) a function defined over a sample space of random experiment is called random variable.

Eg:- 1.) Tossing of 4 coins




2.) Rolling of 2 dice




## ★ Types of random variable:-

There are two types of random variable

1.) Discrete random variable 2.) Continuous random variable.

1.) **Discrete random variable:-** If a random variable takes only countable number of values maybe finite or infinite is called discrete random variable.

Eg:- Number of balls in a cricket over.

## 2) Continuous random variables:-

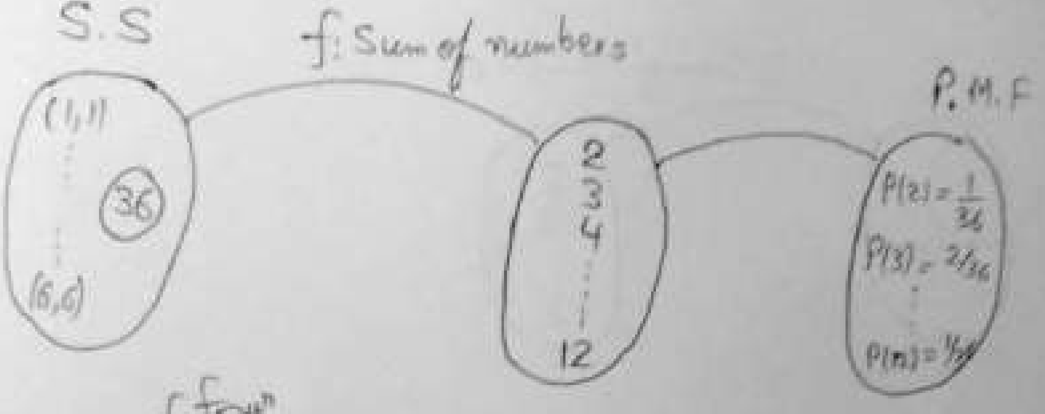
If a random variable takes all possible values between any two real numbers is called continuous random variable

Eg:- Speedometer in bike, Temperature in a city

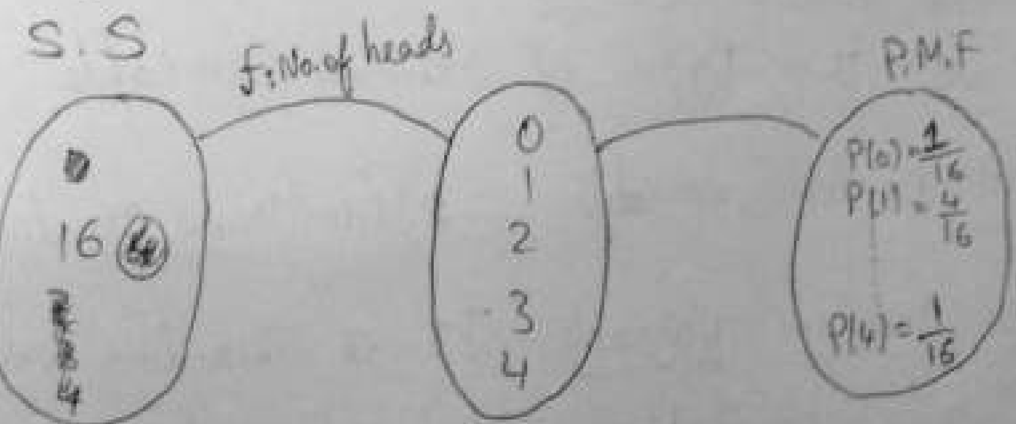
## * Probability Mass function:-

Probability defined over a discrete random variable is called probability mass function

Eg:- Toss of 2 dice




Eg:- Tossing of four coins



i) Probability of every discrete random variable is $>0$ i.e $P(x)>0$

2) Sum of probabilities of all random variables = 1 i.e

$\sum P(x)=1$

* Probability density function:-

A probability defined over a continuous random variable is called probability density function.

i) $P(x)>0$

ii) $\int_{-\infty}^{\infty} f(x)d(x) = 1$

and the corresponding prob.

$P(x)=P(x_0)P(x_1)\ldots\ldots$

* Mathematical Expectation or Expected Value:-

Let $X$ be a discrete random variable, $X = x_0, x_1, x_2, \ldots\ldots \infty$

then M.E (or) Expected value is defined as

$$E(X) = \sum_{i=0}^{\infty} X_i . P(x_i)$$

Similarly, Let $X$ be a continuous random variable, $X = f(x_i)$ where $i = -\infty$ to $\infty$ and $P(x) = P(x_i)$ where $i = -\infty$ to $\infty$ then M.E is defined as

$$E(x) = \int_{-\infty}^{+\infty} x_i . P(x_i)\, dx$$

1.) Find Mathematical Expectation of tossing of a die

Sol:-

| X | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| P(x) | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |

$$E(x) = \sum x_i . P(x_i) = 1.\frac{1}{6} + 2.\frac{1}{6} + 3.\frac{1}{6} + 4.\frac{1}{6} + 5.\frac{1}{6} + 6.\frac{1}{6}$$

$$= \frac{21}{6}$$

2) Find M.E of sum of the numbers on 2 dice

Sol:-

| X | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| P(x) | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

$$E(x) = \sum x_i . P(x_i) = \frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12)$$

$$= \frac{252}{36} = \frac{28}{4} = 7$$

**Note:-** Expected value is nothing but mean value of that random variable.

3) A random variable $X$ has the following probability function

| $X$ | $P(x)$ |
|---|---|
| 0 | 0 |
| 1 | $k$ |
| 2 | $2k$ |
| 3 | $2k$ |
| 4 | $3k$ |
| 5 | $k^2$ |
| 6 | $2k^2$ |
| 7 | $7k^2+k$ |

Find i) Value of $k$ ii) Find $P(X<6)$ iii) $P(0<X<5)$

Sol:- i) $\Sigma P(x) = 1$

$$0 + k + \overset{2k+}{\cancel{2}}(2k) + 3(k) + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k = 1$$

$$k = \frac{1}{10}, -1 \quad \text{But } \boxed{k \neq -1}$$

$$\therefore k = \frac{1}{10}$$

ii) $P(X<6) = P(0) + P(1) + \dots + P(5)$

$$= 0 + k + 2k + 2k + 3k + k^2$$

$$= k^2 + 8k$$

$$= \frac{1}{100} + \frac{8}{10} = \frac{81}{100}$$

iii) $P(0<X<5) = P(1)+P(2)+P(3)+P(4)$

$= k + 2k + 2k + 3k = 8k = \frac{8}{10}$

iv) Find min. value of a such that $P(X \le a) > \frac{1}{2}$

Sol:- $P(0) = 0 > \frac{1}{2}$ ×

$P(0)+P(1) = 0 + \frac{1}{10} = \frac{1}{10} > \frac{1}{2}$ ×

$P(0)+P(1)+P(2) = 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10} > \frac{1}{2}$ ×

$P(0)+P(1)+P(2)+P(3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} > \frac{1}{2}$ ×

$P(0)+P(1)+P(2)+P(3)+P(4) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} > \frac{1}{2}$ ✓

$\therefore \boxed{a=4}$

4.) The diameter of an electric bulb is assumed to be continuous function with Prob. density function $f(x) = 6x(1-x)$, $0 \le x \le 1$. Check that

i) $f(x)$ is prob. density function

Sol:- i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\therefore \int_0^1 6x(1-x) = 6\int_0^1 x - x^2 = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= \frac{6}{6}\left[3x^2 - 2x^3\right]_0^1 = 1[3-2] = 1$

As $\int_0 f(x) = 1$, it is a prob density function

ii) Find the value of b such that $P(x<b) = P(x>b)$

$$\int_0^b f(x)dx = \int_b^1 f(x)dx$$

$$\int_0^b (6x-6x^2)dx = \int_b^1 (6x-6x^2)dx$$

$$\left[3x^2-2x^3\right]_0^b = \left[3x^2-2x^3\right]_b^1$$

$$3b^2-2b^3 = (3-2)-(3b^2-2b^3)$$

$$3b^2-2b^3 = 1-3b^2+2b^3$$

$$4b^3-6b^2+1=0$$

$$\boxed{b = 1/2}$$

5) Find value of k if $f(x) = \begin{cases} ke^{-x/4} & x \geq 0 \\ 0 & x < 0 \end{cases}$

Sol. $\int_0^\infty f(x)\,d(x) = 1$

$$\int_0^\infty ke^{-x/4} = 1 \Rightarrow k\left[\frac{e^{-x/4}}{-1/4}\right]_0^\infty = 1$$

$$k\left[0-\frac{1}{-1/4}\right]=1 \qquad \left[\because \int e^{ax}=\frac{e^{ax}}{a}\right]$$

$$k[4]=1 \Rightarrow \boxed{k=\frac{1}{4}}$$

5) If f(x) is Prob density function, find the value of k

$$f(x)=\begin{cases} k(1-x^2) & 0\leq x\leq 1 \\ 0 & \end{cases}$$

Sol:
$$\int_0^1 k(1-x^2)=1$$

$$\int_0^1 k-kx^2=1 \Rightarrow \left[kx-k\frac{x^3}{3}\right]_0^1=1$$

$$\Rightarrow k-\frac{k}{3}=1 \Rightarrow \boxed{k=\frac{3}{2}}$$

6.) Find the value of a if $f(x)=\frac{3}{4}(x^2+1)$ $0\leq x\leq 1$ and $P(x\leq a)=P(x>a)$

Sol:
$$\int_0^a \frac{3}{4}(x^2+1)=\int_a^1 \frac{3}{4}(x^2+1)$$

$$\cancel{\frac{3}{4}}\left[\frac{x^3}{3}+x\right]_0^a=\cancel{\frac{3}{4}}\left[\frac{x^3}{3}+x\right]_a^1$$

$$\Rightarrow \frac{a^3}{3}+a=\left(\frac{1}{3}+1\right)-\left(\frac{a^3}{3}+a\right)$$

$$\frac{2a^3}{3} + 2a - \frac{4}{3} = 0$$

$$2a^3 + 6a - 4 = 0$$

$$a^3 + 3a - 2 = 0$$

$$a = 0.59$$

7) A random variable X has following probability function. Find the value of k as well as mean value.

| X | P(x) |
|---|---|
| -2 | 0.1 |
| -1 | k |
| 0 | 0.2 |
| 1 | 2k |
| 2 | 0.3 |
| 3 | k |

Sol:- $\sum P(x) = 1$

$$0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$4k + 0.6 = 1 \Rightarrow k = \frac{0.4}{4} = \frac{1}{10} = 0.1$$

$$E(x) = \sum x_i P(x_i) = (-2 \times 0.1) + \left(-1 \times \frac{1}{10}\right) + 0 + 1(0.2) + 2(0.3) + 3(0.1)$$

$$= -\cancel{0.2} - 0.1 + \cancel{0.2} + 0.6 + 0.3$$

$$= 0.8$$

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## ☆ Properties of Mathematical Expectation:-

1.) If $a$ is a constant value then

$E(a) = a$

Proof: $E(X) = \sum x . P(X)$

$= \sum a . P(X) = a(1) = a$

2.) $E(ax) = a . E(x)$ → Proof: $E(ax) = \sum aX . P(x)$

$= a \sum X . P(x)$

$= a E(X)$

3.) $E(X^2) = \sum x^2 . P(x)$

→ If $X, Y$ are any two random variables then Show that $E(X+Y) = E(X) + E(Y)$

Sol:- Let $X, Y$ be two continuous random variables then

$$E(X+Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) f(x,y)\, dx\, dy \qquad \left[\because E(x) = \int_{-\infty}^{\infty} x . f(x)\right.$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\, dx\, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left[ \int_{-\infty}^{\infty} f(x,y)\, dy \right] dx + \int_{-\infty}^{\infty} y \left[ \int_{-\infty}^{\infty} f(x,y)\, dx \right] dy$$

$$= \int_{-\infty}^{\infty} x . f(x) dx + \int_{-\infty}^{\infty} y . f(y) dy$$

$$= E(X) + E(Y)$$

Q) A coin is tossed until a head appears. What is the $\underbrace{\text{expectation}}_{\text{(mean)}}$ of no. of tosses required?

Sol:- Experiment:- Tossing of coin
Event:- Number of tosses

| X | 1 | 2 | 3 | 4 | ........ $\infty$ |
|---|---|---|---|---|---|
| P(x) | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | |

$E(x) = \sum x_i P(x_i)$

$E(x) = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + \ldots\ldots \infty$ — ①

Multiply on both sides with $\frac{1}{2}$

$\frac{1}{2} E(x) = 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{16} + 4 \times \frac{1}{32} + \ldots \infty$ — ②

① − ②

$\frac{1}{2} E(x) = 1 \times \frac{1}{2} + \cancel{2\times}\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \infty$

This is an Geometric series, $r = \frac{1}{2}$

$\frac{1}{2} E(x) = \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1 \Rightarrow \boxed{E(x) = 2}$

a) If $f(x) = \begin{cases} a+bx & 0<x<1 \\ 0 & \text{else} \end{cases}$

If mean value is $1/3$, find a and b

Sol:- We know that $\int_0^1 f(x) = 1$

$$\Rightarrow \int_0^1 (a+bx)dx = 1$$

$$\left[ax + \frac{bx^2}{2}\right]_0^1 = 1$$

$$a + \frac{b}{2} = 1$$

$$2a+b = 2 \quad — \text{(1)}$$

Given,

$$E(x) = \text{mean} = \frac{1}{3}$$

$$\int_0^1 x.f(x)dx = \frac{1}{3}$$

$$\int_0^1 x(a+bx)dx = \frac{1}{3}$$

$$\int_0^1 (ax+bx^2)dx = \frac{1}{3}$$

$$\left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\frac{a}{2} + \frac{b}{3} = \frac{1}{3}$$

$$3a+2b = 2 \quad — \text{(2)}$$

Solving ① & ② we get

$3a+2(2-2a)=2$

$3a+4-4a=2 \Rightarrow \boxed{a=2}$

$b=2-2a$

$=2-2(2)=2-4=-2 \quad \boxed{b=-2}$

Q)

| $x$ | $P(x)$ |
|---|---|
| -3 | 0.05 |
| -2 | 0.1 |
| -1 | 0.3 |
| 0 | 0 |
| 1 | 0.3 |
| 2 | 0.15 |
| 3 | 0.1 |

Find $E(x)$, $E(x^2)$, $E(2+3x)$

Sol. $E(x)=\sum_{i=-3}^{3} x_i P(x_i) = (-3)(0.05)+(-2)(0.1)+(-1)(0.3)$

$+0+1(0.3)+2(0.15)+3(0.1)$

$=-0.15-0.2+0.3+0.3$

$=0.6-0.35=0.25=1/4$

$E(x^2)=\sum_{i=-3}^{3}(x_i^2).P(x_i) = 9(0.05)+4(0.1)+1(0.3)+0+1(0.3)$

$+4(0.15)+9(0.1)$

$=0.45+0.4+0.3+0.3+0.6+0.9$

$=2.95$.

$E(2+3x)=E(2)+E(3x)$

$=2+3E(x)=2+\frac{3}{4}=\frac{11}{4}$

Q) A Man plays a game with cards numbered 1-100. He will get Rs 40 if he gets a card with number 60 or greater. He has to pay Rs. 50 otherwise. Do you suggest him to play the game

Sol:- Exp:- Selecting a card

Events

$P(1-59) = \frac{59}{100}$ $\quad P(60-100) = \frac{41}{100}$

$$E(x) = \left(\frac{41}{100} \times 40\right) + \left(\frac{59}{100} \times -50\right)$$

Since mean is negative, we suggest him to not play the game.

☆ Variance:- $V(x) = E(x^2) - (E(x))^2$

$$= E(x - \bar{x})^2$$

Q) If a is a constant value, show that $V(a) = 0$

Sol:- $V(x) = E(x^2) - (E(x))^2$

$V(a) = E(a^2) - (E(a))^2$

$= a^2 - a^2$

$= 0$

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Q) Show that $V(ax) = a^2 V(x)$

Sol:- $V(ax) = E(a^2x^2) - (E(ax))^2$

$= a^2 E(x^2) - a^2 E(x)^2$

$= a^2 (E(x^2) - E(x)^2)$

$= a^2 (V(x))$

$\Rightarrow V(ax+b) = a^2 V(x)$

) Find the value of b such $P(X>b) = 0.05$
where $f(x) = 3x^2 \quad 0<x<1$

Sol:- $\int_b^1 f(x)\,dx = 0.05$

$\int_b^1 3x^2\,dx = 0.05 \Rightarrow [x^3]_b^1 = 0.05$

$\Rightarrow 1 - b^3 = 0.05$

$\Rightarrow b^3 = 0.95$

$\Rightarrow b = 0.983$ //

Unit-II

* **Binomial distribution:-**

Let a random experiment be performed repeatedly and let occurance of an event be called a success & non-occurrance is called a failure, then to know or evaluate 'r' successes, if in conduction of that experiment, number of we apply binomial distribution and it is defined as follows:

**Definition of binomial distribution:-**

A random variable X is set to follow binomial distribution if it assumes only non-negative integers & its probability mass function is defined as $P(X=x) = nC_x (p)^x (q)^{n-x}$ where $x = 0, 1, 2, \ldots n$ where $n$ & $p$ are parameters of binomial distribution where $n$ represents repetation of random experiment of $n$ times. & $p$ represents prob of success (or) prob. of happening of an event and $q = 1-p$ i.e prob. of not happening of that event.

Q) 10 coins are tossed simultaneously. Find the probability of getting atleast 7 heads.

Sol:- Random experiment = Toss of a coin

$n = 10$ (repeating tossing of a coin 10 times)

$P$ = prob. of success i.e getting a head.

$P = \frac{1}{2}$ $\quad q = \frac{1}{2} = 1 - p$

$$P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$$

$$= 10_{C_7}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7 + 10_{C_8}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^8 + 10_{C_9}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^9$$

$$+ 10_{C_{10}}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10} \times \left(10_{C_7} + 10_{C_8} + 10_{C_9} + 10_{C_{10}}\right)$$

Q) A and B play a game in which their chances of winning ~~is~~ are in the ratio 3:2. Find A's chance of winning atleast 3 games out of 5 games played.

Sol:- R.E = Playing of game between A and B

$n = 5$

$P$ = prob. of success = prob. of A winning $= \frac{3}{5}$

$q = \frac{2}{5}$

$P(X \geq 3) = P(3) + P(4) + P(5)$

$$= {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 + {}^5C_4 \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right)^1 + {}^5C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^0$$

$=$

Q.) A MCQ test consists of 8 questions with 3 answers for each A student answers each question by rolling a die & checking (answering) the first answer if he gets 1 or 2, the second answer if he gets 3 or 4 and third answer if he gets 5 or 6. To get distinction, the student must secure atleast 75% correct answers. If there is no negative marking what is the prob that a student secures distinction.

Sol:- $n = 8$

$p$ = prob. of success $= \frac{1}{3}$  $q$ = prob. of failure $= 1 - p$

$= 1 - \frac{1}{3} = \frac{2}{3}$

$P(X \geq 6) = P(6) + P(7) + P(8)$

$$= {}^8C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^2 + {}^8C_7 \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^1 + {}^8C_8 \left(\frac{1}{3}\right)^8 \left(\frac{2}{3}\right)^0$$

Q) 7 coins are tossed and number of heads are noted. The experiment is repeated 128 times & the following distribution is obtained. Find frequency by using binomial distribution (or) fit a binomial distribution

| No of head (x) | frequency f(x) |
|---|---|
| 0 | 7 |
| 1 | 6 |
| 2 | 19 |
| 3 | 35 |
| 4 | 30 |
| 5 | 23 |
| 6 | 7 |
| 7 | 1 |

Sol. $n = 7$ $\qquad N = 128$

$p = \frac{1}{2} \quad q = \frac{1}{2}$

$$P(0) = {}^{n}C_0 (p)^{0} q^{n-0} = {}^{7}C_0 \left(\frac{1}{2}\right)^{0} \left(\frac{1}{2}\right)^{7}$$

$$= 1 \times \left(\frac{1}{2}\right)^{7} = \frac{1}{128}$$

$$N \times P(0) = 128 \times \frac{1}{128} = 1$$

$$P(1) = {}^{n}C_1 (p)^{1} q^{n-1} = {}^{7}C_1 \left(\frac{1}{2}\right)^{1} \left(\frac{1}{2}\right)^{6} = \frac{7}{128}$$

$$\text{Expected freq} = N \times P(1) = 128 \times \frac{7}{128} = 7$$

$$P(2) = {}^7C_2 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^2 = \frac{7.6}{1.2}\left(\tfrac{1}{2}\right)^7 = 21 \times \frac{1}{128}$$

$$N \times P(2) = \cancel{128} \times \frac{21}{\cancel{128}}$$

$$P(3) = {}^7C_3 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^3 = \frac{7.\cancel{6}.5}{1.\cancel{2}.\cancel{3}}\left(\tfrac{1}{2}\right)^7 = \frac{35}{128}$$

$$N \times P(3) = \cancel{128} \times \frac{35}{\cancel{128}} = 35$$

$$P(4) = {}^7C_4 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^4 = \frac{7.\cancel{6}.5.\cancel{4}}{1.\cancel{2}.\cancel{3}.\cancel{4}}\left(\tfrac{1}{128}\right) = \frac{35}{128}$$

$$N \times P(4) = 128 \times \frac{35}{128} = 35$$

$$P(5) = {}^7C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^2 = \frac{7.6.\cancel{5}.\cancel{4}.\cancel{3}}{1.2.\cancel{3}.\cancel{4}.\cancel{5}} = \frac{21}{128}$$

$$N \times P(5) = 21$$

$$P(6) = {}^7C_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^1 = \frac{7}{128} \quad ; \quad N \times P(6) = 7$$

$$P(7) = {}^7C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^0 = \frac{1}{128} \quad ; \quad N \times P(7) = \frac{1}{128} \times 128 = 1$$

| No. of heads ($x$) | freq $f(x)$ | freq by using B.D |
|---|---|---|
| 0 | 7 | 1 |
| 1 | 6 | 7 |
| 2 | 19 | 21 |
| 3 | 35 | 35 |
| 4 | 30 | 35 |
| 5 | 23 | 21 |
| 6 | 7 | 7 |
| 7 | 1 | 1 |

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## ☆ Applications of Binomial distribution

1.) Binomial distribution can be applied to the following experiments

a) No. of repetation of experiment is known

b) prob. of success or failure of each trial is fixed.

2.) We can apply B.D such as tossing of a coin, no. of hitting of a target, rolling of a die etc.

## ☆ Poission distribution:-

Note:- is a limiting case of binomial distribution where $n \to \infty$

Definition:-

A random variable X is said to follow Poission distribution if it assumes only non-negative values & its prob mass function is given by

$$P(x,\lambda) = P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad x = 0,1,2,\dots\infty$$

where $\lambda$ is known as parameter of Poission distribution and it is also known as average value of Poission distribution

$$\boxed{\lambda = np}$$

where $n$ is no. of repeatation of experiment

$p \Rightarrow$ prob. of success

* Application of Poission distribution

1.) Poission distribution can be applied to follow

a) No. of suicides in a city in a day.

b) No. of defective blades in a packet.

c) No. of air accidents in a unit of time in a country.

d) No. of telephone calls received at a particular telephone exchange in some unit of time

e) No. of heart-attacks, snake-bites etc

1) A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pin in a box of 100 and guarantees that not more than 10 pins will be defective, what is the prob. that a box will fail to meet its guaranteed quality.

Sol:- $n = 100$

$$\lambda = \frac{5}{100} = 0.05$$

$$P(0) = \frac{e^{-\lambda} \lambda^{x}}{x!} = \frac{e^{-0.05} (0.05)^{0}}{0!} = 0.951$$

$$P(x > 10) = 1 - P(x \le 10)$$

$$= 1 - \sum_{x=0}^{10} \frac{e^{-\lambda} \lambda^{x}}{x!}$$

prob that a box will fail to meet its guaranteed quality

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2) Six coins are tossed 6400 times. Using Poission distribution, find the approximate prob. of getting 6 heads 6 times

Sol: $\lambda = np = 6400 \times \left(\frac{1}{2}\right)^6$ $n = 6400$

$= 100$ $p = \left(\frac{1}{2}\right)^6$

$$P(x=6) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-100} (100)^6}{6!}$$

$$= 5.166 \times 10^{-35}$$

3) In a book of 520 pages, 390 typographical errors occur. Assuming Poission law for no. of errors per page, find the prob. that a random sample of 5 pages will have no error.

Sol: $\lambda$ = avg. no. of mistakes per page

$$= \frac{390}{520} = \frac{3}{4} = 0.75$$

prob of zero error in 1 page

$$P(0) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.75} (0.75)^0}{0!} = 0.47$$

Prob. of zero errors in 5 pages $= (P(0))^5 = (0.47)^5$

$= 0.0235$

4) Fit a poission distribution for the following data

| x | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| freq | 109 | 65 | 22 | 3 | 1 |

Sol:- $\lambda = Avg = \frac{\sum x_i f_i}{\sum f_i} = \frac{122}{200} = 0.61$

$P(0) = \frac{e^{-\lambda} \lambda^0}{0!} = \frac{e^{-0.61} \times 1}{1} = 0.543$

$N \times P(0) = 200 \times 0.543 = 108.6 \simeq 109$

$N \times P(1) = 200 \times \frac{e^{-0.61} \times (0.61)^1}{1!} = 66.28 \simeq 66$

$N \times P(2) = 200 \times \frac{e^{-0.61} \times (0.61)^2}{2!} = 20.21 \simeq 20$

$N \times P(3) = 200 \times \frac{e^{-0.61} \times (0.61)^3}{3!} = 4.11 \simeq 4$

$N \times P(4) = 200 \times \frac{e^{-0.61} \times (0.61)^4}{4!} = 0.626 \simeq 1$

| x | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| freq | 109 | 65 | 22 | 3 | 1 |
| PD | 109 | 66 | 20 | 4 | 1 |

5). After correcting 50 pages of the proof of a book, the proof reader finds that on average there are 2 mistakes per 5 pages. How many pages would you expect to find with 0, 1, 2, 3 and 4 errors in 1000 pages respectively.

Sol: $\lambda = \frac{2}{5} = 0.4$ , $N = 1000$

$$N \times P(0) = 1000 \times \frac{e^{-\lambda}\lambda^0}{0!} = 1000 \times e^{-0.4} \times 1 = 670.32 \simeq 670$$

$$N \times P(1) = \frac{1000 \times e^{(-0.4)} \times (0.4)^1}{1!} = 268.1 \simeq 268$$

$$N \times P(2) = 1000 \times \frac{e^{-0.4}(0.4)^2}{2!} = 53.6 \simeq 54$$

$$N \times P(3) = 1000 \times \frac{e^{-0.4} \times (0.4)^3}{3!} = 7.15 \simeq 7$$

$$N \times P(4) = 1000 \times \frac{e^{-0.4}(0.4)^4}{4!} = 0.71 \cong 1.$$

| X | frequency |
|---|---|
| 0 | 670 |
| 1 | 268 |
| 2 | 54 |
| 3 | 7 |
| 4 | 1 |

6) If $x$ is a poission variable such that $P(x=2) = 9 \times P(x=4) + 90 \times P(x=6)$. Find value of $\lambda$ as well as mean value of variable $x$

Sol:-

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9 \times \frac{e^{-\lambda}\lambda^4}{4!} + 90 \times \frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{\cancel{e^{-\lambda}\lambda^2}}{2} = 9\cancel{e^{-\lambda}\lambda^2}\left[\frac{1}{4!} + \frac{10\lambda^4}{6!}\right]$$

$$\frac{1}{2} = 9\left[\frac{1}{24} + \frac{\lambda^4}{72}\right]$$

$$\frac{1}{18} - \frac{1}{24} = \frac{\lambda^4}{72} \Rightarrow \frac{\lambda^4}{72} = \frac{1}{72}$$

$$\lambda^4 = \frac{72}{72} = 1$$

$$\lambda = 1, -1, i, -i$$

$$\therefore \boxed{\lambda = 1}$$

Mean value is also 1.

7) A manufacturer, who produces medicine bottles finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 1000 boxes from the producer of bottles. Using P.D, find how many boxes will contain i) no defective ii) atleast ~~one~~ two defective

Sol:- $\lambda = np = 500 \times \frac{0.1}{100} = 0.5$

↓

Avg. no of defective bottles in a box which contains 500 bottles

$$P(X=0) = \frac{e^{-\lambda} \lambda^0}{0!} = \frac{e^{-0.5} \times 1}{1} = 0.606$$

$N = 1000$

i) $N \times P(0) = 1000 \times 0.606 = 606$ ~~bottles~~

↓

boxes that will contain no defective

ii) $P(X \geq 2) = 1 - P(X<2)$

$= 1 - [P[0] + P(1)]$

$= 1 - [0.606 + 0.303] = 1 - 0.909$

$= 0.091$

$0.091 \times 1000 = 91$ boxes

* Moments

Let X be a discrete random variable and P(x) be its prob mass function.

We know that $E(x) = \sum_{i=0}^{n} x_i \times p(x_i)$

Let A be a point then and X be a discrete random variable

$r^{th}$ moment about point A of random variable X is given by

$$\mu_r' = E\left[(X-A)^r\right]$$

$$= \sum_{i=0}^{n} (x_i - A)^r \cdot P(x)$$

If A=0 then $r^{th}$ moment about origin

$$\mu_r = E(X^r) = \sum x^r \times P(x)$$

Put r=0

$$\mu_0' = \sum x^0 \times P(0) = \sum P(0) = 1$$

$$\boxed{\mu_0' = 1}$$

Put r=1

$$\mu_1' = E(x) = \text{mean} = \sum x \times p(x)$$

$\mu_2' = E(x^2) = \sum x^2 p(x)$

$\mu_3' = E(x^3) = \sum x^3 . P(x)$

$\mu_4' = E(x^4) = \sum x^4 . P(x)$

If $A = \bar{x}$ = mean then $r^{th}$ moment is given by

$\mu_r = E\left[(x-\bar{x})^r\right]$

[Moments about mean]
↓
Central moments

**$r=0$**

$\mu_0 = E\left[(x-\bar{x})^0\right]$

$= \sum (x-\bar{x})^0 \times P(x) = \sum P(x) = 1$

**$r=1$**

$\cancel{\mu_1 = \sum (x-\bar{x})^1 P(x)}$

$\cancel{\mu_2 = E\left[(x-\bar{x})^2\right] = \sum (x-\bar{x})^2 . P(x)}$

$\mu_1 = E\left[(x-\bar{x})^1\right]$

$= E(x) - E(\bar{x}) = E(x) - \bar{x} = \bar{x} - \bar{x} = 0$

$\mu_2 = E\left[(x-\bar{x})^2\right]$ = variance

$= E(x^2) - (E(x))^2$

$\mu_3 = E\left[(x-\bar{x})^3\right] = \sum (x-\bar{x})^3 \times p(x)$

$\mu_4 = E\left[(x-\bar{x})^4\right] = \sum (x-\bar{x})^4 . p(x)$

## * Moment Generating Function:-

A function which generates moments is called moment generating function and is given as $M_x(t) = E(e^{tx})$

$$= E\left(1 + tx + \frac{t^2}{2!}x^2 + \frac{t^3x^3}{3!} + \dots\right) \quad \left[\because e^x = 1 + x + \frac{x^2}{2!} + \dots\right]$$

$$= E(1) + tE(x) + \frac{t^2}{2!}E(x^2) + \dots$$

$$= \mu_0' + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

$$\frac{d}{dt}\left[M_x(t)\right]_{t=0} = \mu_1' \qquad \left[\frac{d^2}{dt^2}M_x(t)\right]_{t=0} = \mu_2'$$

$$\frac{d^3}{dt^3}\left[M_x(t)\right]_{t=0} = \mu_3' \qquad \left[\frac{d^4}{dt^4}M_x(t)\right]_{t=0} = \mu_4'$$

If the function is not expandable, then differenciate on both sides as required to get $\mu_1'$, $\mu_2'$,

## * Relation between moment about origin and moment about mean

$\mu_1' = E(x) \rightarrow$ mean $\qquad \mu_1 = E(x-\bar{x})$

$\mu_2' = E(x^2) \qquad \mu_2 = E(x^2) - (E(x))^2 = E(x-\bar{x})^2 =$ variance

$\mu_3' = E(x^3) \qquad \mu_3 = E(x-\bar{x})^3$

* 1) $\boxed{\mu_1 = 0}$

2) $\mu_2 = E(x^2) - (E(x))^2$

$= \mu_2' - (\mu_1')^2$

3.) $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

4.) $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

→ <u>Skewness</u> – Skewness is also known as lack of symmetry. It discusses symmetrical nature of the given data. It is given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

If $\beta_1 = 0$, curve is symmetrical

$\beta_1 > 0$, +ve side avg.

$\beta_1 < 0$, −ve side avg.

→ <u>Kurtosis</u>:-

Kurtosis defines flatness of the peak of given curve. It is defined as

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

# ☆ Moment generating function of Binomial Distribution

We know that

$$M_x(t) = E(e^{tx})$$

$p(x) \to$ probability mass function

$$= \sum e^{tx} \cdot p(x)$$

$[\because p(X=x) = {}^nC_x p^x q^{n-x}]$

$$= \sum e^{tx} \cdot {}^nC_x (p)^x (q)^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (q)^{n-x} (e^t p)^x$$

$$= (q + e^t p)^n$$

$[\because (a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b^1 + \cdots {}^nC_n a^0 b^n]$

$[\text{Here } a = e^t p, \ b = q]$

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}(q + e^t p)^n\right]_{t=0}$$

$$= \left[n(q + e^t p)^{n-1}(0 + e^t p)\right]_{t=0}$$

$$= \left[n(q + e^0 p)^{n-1}(e^0 p)\right]$$

$$= \left[n(p+q)^{n-1} p\right] = np$$

$[\because \text{In B.D } p+q=1]$

$$\mu_1' = E(X) = \text{mean} = np$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}(pe^t+q)^n\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[n(pe^t+q)^{n-1}(e^t p)\right]\right]_{t=0}$$

$$= \cancel{n(e^t p)}\left[np\,\frac{d}{dt}\left[(pe^t+q)^{n-1}(e^t)\right]\right]_{t=0}$$

$$= np\left[\left[(pe^t+q)^{n-1}\cdot e^t + e^t(n-1)(pe^t+q)^{n-2}\cdot e^t p\right]\right]_{t=0}$$

$$= np\left[(p+q)^{n-1}\cdot 1 + 1(n-1)(p+q)^{n-1}\cdot 1.p\right]$$

$$= np\left[1+(n-1)p\right] = np\left[1+pn-p\right]$$

$$= np + n^2p^2 - np^2$$

$$\boxed{\mu_2' = np+n^2p^2-np^2}$$

$$\rightarrow \mu_3' = \left[\frac{d^3}{dt^3} M_x(t)\right]_{t=0} = \left[\frac{d^3}{dt^3}(pe^t+q)^n\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(npe^t(pe^t+q)^{n-1} + np^2(e^t)^2(n-1)(pe^t+q)^{n-2}\right]_{t=0}$$

$$= np\left[\left[\frac{d}{dt}\left(e^t(pe^t+q)^{n-1} + p.e^{2t}(n-1)(pe^t+q)^{n-2}\right)\right]\right]_{t=0}$$

## * Central moments of Binomial distribution

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= (np + n^2p^2 - np^2) - n^2p^2$

$= np - np^2 = np(1-p) = npq$ = variance

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np - 3[np + n(n-1)p^2]n$

$\beta_1 =$ $+ 2(np)^3$ $= \boxed{npq(q-p)}$

Q) If mean & variance of B.D are 3 and 4 Comment on it

Sol:- Mean $= np = 3$

Variance $= npq = 4$

$$\frac{\cancel{np}q}{\cancel{np}} = \frac{4}{3} \Rightarrow q = \frac{4}{3}$$

$\because q$ is probability, it can't be $>1$.

Therefore mean & variance can't be 3 & 4 resp.

Q) If mean & variance of B.D are 4 and $\frac{4}{3}$ Find $P(X \geq 1)$

Sol:- $np = 4 \quad npq = \frac{4}{3}$

$$q = \frac{\frac{4}{3}}{4} = \frac{1}{3} \qquad p = 1 - \frac{1}{3}$$

$$p = \frac{2}{3}$$

$\therefore \quad np = 4$

$n = \frac{4}{p} = \frac{4}{2/3} = 6$

$\therefore P(x \geq 1) = 1 - P(x < 1)$

$= 1 - P(0)$

$= 1 - 6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^6$

$= 1 - 1.1.\left(\frac{1}{3^6}\right) = 1 - \frac{1}{3^6}$

$= 0.998$

Q) Find mean & variance of Poission distribution

or Show that mean & variance of P.D is '$\lambda$'!

Sol:- $M_x(t) = E(e^{tx})$

$= \sum_{n=0}^{\infty} e^{tx} . p(x)$

$= \sum e^{tx} . \frac{e^{-\lambda} \lambda^x}{x!}$

$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$

$= e^{-\lambda} . (e^{\lambda e^t})$ $\quad \left[\because e^x = 1 + x + \frac{x^2}{2!} + \dots\right]$

$= e^{\lambda(e^t - 1)} \rightarrow$ MGF for P.D

$$\text{Mean} = E(x) = \mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt} e^{\lambda(e^t-1)}\right]_{t=0}$$

$$= \left[e^{\lambda(e^t-1)} \cdot (\lambda e^t)\right]_{t=0}$$

$$= e^{\lambda(1-1)} \cdot \lambda e^0$$

$$= e^0 \cdot \lambda \cdot 1 = 1 \cdot \lambda \cdot 1 = \lambda$$

$$\mu_2' = \left[\frac{d^2}{dt^2}(M_x(t))\right]_{t=0}$$

$$= \left[\frac{d}{dt} e^{\lambda(e^t-1)} \cdot \lambda e^t\right]_{t=0} = \lambda\left[e^{\lambda(e^t-1)} \cdot e^t + e^t \cdot e^{\lambda(e^t-1)} \lambda e^t\right]_{t=0}$$

$$= \lambda\left[e^{\lambda(0)} \cdot e^t + e^t \cdot e^{\lambda(0)} \cdot \lambda e^t\right]_{t=0}$$

$$= \lambda\left[e^t + \lambda e^t \cdot e^t\right]_{t=0}$$

$$= \lambda\left[e^t(\lambda e^t + 1)\right]_{t=0}$$

$$= \lambda\left[1(\lambda \cdot 1 + 1)\right] = \lambda^2 + \lambda$$

$$\therefore \text{Variance} = \mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2 = \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

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Q) A man pays Rs.10 to hit a target. The prob. of hitting a target is 0.4. He gets Rs.20 if he hits a target. What is expected value?

Sol.- $E(x) = \sum x.p(x)$

| $x$ | 10 | -10 |
|---|---|---|
| $P(x)$ | 0.4 | 0.6 |

$E(x) = 10(0.4) + (-10)(0.6)$
$= 4 - 6 = -2$

Q) Three friends go to a restaurant & write their names on a piece of paper & mix. If anybody picks his own name, he has to pay bill for others. If nobody picks their own name the game will be repeated. What is the prob. neither of these situation occur?

Sol:-
1.) A B C - All will pay their own bill
2.) A C B - one will pay
3.) B A C - one
4.) B C A - none will pay
5.) C A B - none
6.) C B A - one

$P(\text{none}) = \frac{2}{6} = \frac{1}{3}$

$P(\text{one}) = \frac{3}{6} = \frac{1}{2}$

$P(\text{All}) = \frac{1}{6}$

Q) A box contains 4 blue & 5 green balls. What is the prob of getting 4 balls of which 2 green, 2 blue? i) without replacement ii) with replacement

Sol:- i) $P(4 \text{ balls}) = \dfrac{{}^4C_2 \times {}^5C_2}{{}^9C_2}$

$=$

ii) $P(4 \text{ balls}) = \left(\dfrac{4\times4\times5\times5}{9\times9\times9\times9}\right)\times 6$

Q) A pack of cards is distributed among 4 players. What is the prob of getting ~~only~~ one ace by first player

Sol:- $P(E) = \dfrac{\overset{\text{getting one ace}}{{}^4C_1} \times \overset{\text{rest 12 cards}}{{}^{48}C_{12}}}{{}^{52}C_{13}}$

Q) A mall have 4 floors and the shops on each floor as given below:

| floor | Shop |
|---|---|
| $I^{st}$ | Coffee |
| II | Garment |
| III | Theater |
| I | Garment |
| II | Food court |
| III | Games |
| IV | Book |

Q) A man visits to mall, what is the prob he goes to either garment shop or second floor?

Sol:- A = visits garment shop

B = visiting second floor i.e visiting any shop on 2nd floor

$P(A) = \frac{2}{7}$ , $P(B) = \frac{2}{7}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{2}{7} + \frac{2}{7} - \frac{1}{7}$

$= \frac{3}{7}$

Q) The prob that a thief being caught by police during robbery is ~~only~~ 0.1. What is the prob. that he's being caught in his 4th attempt first time?

Sol:- $n = 4$ , $P = 0.1$ $q = 0.9$

$P(X=4) = {}^{n}C_{x} (P)^{x} (q)^{n-x}$

$= {}^{4}C_{4} (P)^{4} (q)^{0}$

$= (0.1)^{4}$

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Q) Prob. of winning a match is 0.55. What is the prob of winning 3 consecutive games?

Sol:- P(winning 3 consecutive games) = $(0.55)^3$

Q) A Batsman hits a six on an average in every 12 balls. What is the prob. that he hits 4 sixes in 20 balls.

Sol:- $p = \frac{1}{12}$, $q = \frac{11}{12}$, $n = 20$ $x = 4$

$$P(x=4) = {}^{20}C_4 \left(\frac{1}{12}\right)^4 \left(\frac{11}{12}\right)^{16}$$

(or)

$$\lambda = np = 20 \times \frac{1}{12} = \frac{5}{3}$$

$$P(4) = \frac{e^{-\lambda} \lambda^4}{4!} = e$$

# Unit-III

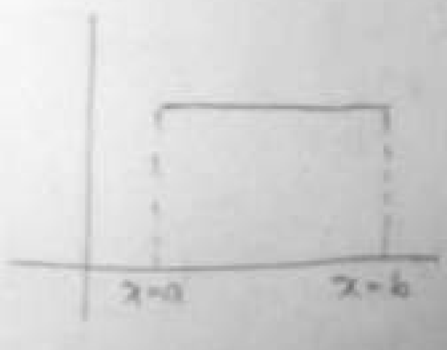
## Continuous Probability distribution

→ **Uniform or rectangular probability distribution:-**

A continuous random variable X is said to follow uniform distribution if its probability density function over an interval $(a,b)$ where $a<b$ is given by

$$f(x,a,b) = P(X) = \begin{cases} \frac{1}{b-a} & \text{for } a<x<b \\ 0 & \text{otherwise} \end{cases}$$

It is also known as rectangular distribution




Here a and b both are parameter of uniform distribution

→ If the given interval is $-a < x < a$

then $f(x,a,b) = P(x) = \begin{cases} \frac{1}{2a} & -a<x<a \end{cases}$

1) If X is uniformly distributed with mean value 1 and variance $\frac{4}{3}$. Find values of a and b as well as $P(X>0)$

Sol: Mean $= \mu_1' = E(X) = \int_a^b x.f(x)dx$

[Mean & variance of Uniform distribution]

$$= \frac{1}{b-a}\int_a^b x.f(x)dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{(b-a)2} = \frac{b+a}{2}$$

Variance is $\mu_2$

W.K.T $\mu_2 = \mu_2' - (\mu_1')^2$

$$\mu_2' = E(x^2) = \int_a^b x^2.f(x)dx = \int_a^b x^2.\frac{1}{b-a}dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+a^2+ab}{3}$$

$\therefore$ variance $= \mu_2 = \mu_2' - (\mu_1')^2$

$$= \frac{b^2+a^2+ab}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4b^2+4a^2+4ab-3b^2-3a^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

Given Mean = 1

$\frac{b+a}{2} = 1 \Rightarrow a+b = 2$ — (1)

Variance $= \frac{4}{3}$

$\frac{(a-b)^2}{\cancel{12}_4} = \frac{4}{\cancel{3}_1} \Rightarrow (a-b)^2 = 16$

$\Rightarrow a-b = \pm 4$ — (2)

$$\begin{array}{r} a+\cancel{b} = 2 \\ a-\cancel{b} = 4 \\ \hline 2a = 6 \end{array}$$

$\boxed{a=3}$ then $b = 2-a$

$b = 2-3 = -1$

$\boxed{b=-1}$

But $a \not> b$

$\therefore$ Solving by taking $a-b=-4$

$$\begin{array}{r} a+\cancel{b} = 2 \\ a-\cancel{b} = -4 \\ \hline 2a = -2 \end{array} \Rightarrow \boxed{a=-1} \quad \boxed{b=3}$$

$\therefore a=-1, \ b=3$

$f(x) = \frac{1}{b-a} = \frac{1}{3+1} = \frac{1}{4}$

$P(X>0) = \int_0^3 f(x).dx$

$= \frac{1}{4}[x]_0^3 = \frac{1}{4}(3-0) = \frac{3}{4}$

Metro trains

2) On a certain line run every half hour between midnight & six in the morning. What is the prob. that a man entering the station at a random time during this period will have to wait atleast 20 minutes?

Sol:- $f(x) = \begin{cases} \frac{1}{30} & \overset{a}{0} < x < \overset{b}{30} \end{cases}$

$$P(20 < x < 30) = \int_{20}^{30} f(x).dx$$

$$= \frac{1}{30}\left[x\right]_{20}^{30}$$

$$= \frac{10}{30} = \frac{1}{3}$$

3) Find moment generating function of uniform distribution

Sol:- $M_x(t) = E(e^{tx})$

$$= \int_a^b e^{tx}.f(x)\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

29/02/2020

## Exponential distribution:-

→ A random variable $x$ is set to follow exponential distribution with parameter $\theta > 0$. If its Probability density function is given by

$$f(x, \theta) = p(x) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

Find moment generating function of Exponential distribution

MGF

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \cdot \theta e^{-\theta x}\, dx$$

$$= \theta \int_0^{\infty} e^{x(t-\theta)}\, dx$$

$$= \cancel{\theta} \cancel{\left[ \frac{e^{x(t-\theta)}}{t-\theta} \right]_0^{\infty}}$$

$$= \theta \int_0^{\infty} e^{-x(\theta - t)}\, dx$$



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$$= \theta \left[ \frac{e^{-x(\theta - t)}}{-(\theta - t)} \right]_0^\infty$$

$$= \frac{\theta}{\theta - t} \left[ 0 - e^{-0} \right]$$

$$= \theta / \theta - t$$

$$M_x(t) = \frac{\theta}{\theta(1 - t/\theta)} = 1/1 - t/\theta$$

$$M_x(t) = \left(1 - t/\theta\right)^{-1}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 \ldots$$

$$\left\{ M_x(t) = 1 + t/\theta + t^2/\theta^2 + t^3/\theta^3 + \ldots \infty \right\}$$

Find mean and variance of exponential distribution.

$$\mu_1' = \text{mean} = E(x) \text{ or } \frac{d}{dt} M_x(t) \Big|_{t=0} \text{ or Coeff of } \frac{t^1}{1!}$$

$$\mu_1' = \text{mean} = 1/\theta$$

$$\mu_2' = \text{Coeff of } \frac{t^2}{2!} = \frac{2}{\theta^2}$$

$$\mu_3' = \text{Coeff of } t^3/3! = 6/\theta^3$$

$$\mu_4' = \text{Coeff of } \frac{t^4}{4!} = 24/\theta^4$$



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$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2$$

$$\mu_2 = \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2$$

$$\left\{ \text{Variance} = \mu_2 = 1/\theta^2 \right\}$$

$$\left\{ \text{Variance} = \frac{\text{mean}}{\theta} \right\}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \frac{6}{\theta^3} - 3\left(\frac{2}{\theta^2}\right)\left(\frac{1}{\theta}\right) + 2\,1/\theta^3$$

$$\mu_3 = \frac{6 - 6 + 2}{\theta^3} = 2/\theta^3$$

$$\boxed{\mu_3 = 2/\theta^3}$$

Skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(2/\theta^3)^2}{(1/\theta^2)^3} = \frac{4/\theta^6}{1/\theta^6} = 4$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= \frac{24}{\theta^4} - 4\left(\frac{6}{\theta^3}\right)(1/\theta) + 6\left(\frac{2}{\theta^2}\right)\left(\frac{1}{\theta^2}\right) - 3\left(\frac{1}{\theta^4}\right)$$



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$$\mu_4 = \frac{9}{\theta^4}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{9/\theta^4}{(1/\theta^4)} = 9$$

## * Normal distribution :-

> A Continuous random variable $X$ is said to follow normal distribution if its probability density function is given by

$$f(x, \mu, \sigma) = P(X = x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$-\infty < x < \infty$$

where $\mu$ & $\sigma$ are parameters of normal distribution

$\mu$ = mean

$\sigma$ = standard deviation

$\sigma^2$ = variance

> Standard normal variable :-

Let $z = \frac{x - \mu}{\sigma}$

where $z$ is standard normal variable

& $x$ is normal variable.

$\mu$, & $\sigma$ are mean & S.D



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$$\left[\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\right] \quad -\infty < z < \infty$$

Mean = 0

S.D = 1 = variance

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad ; \quad \int_{-\infty}^{\infty} \phi(z)\,dz = 1$$

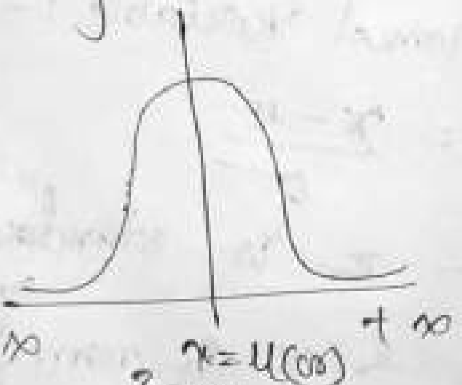
$$\Rightarrow \int_{-\infty}^{\mu} f(x)\,dx = \int_{\mu}^{\infty} f(x)\,dx = 1/2 \quad ; \quad \int_{-\infty}^{0} \phi(z)\,dz = \int_{0}^{\infty} \phi(z)\,dz = 1/2$$

i) Properties of Normal distribution :-

i) Probability density function

$$f(x,\mu,\sigma) = P(X = x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

ii) The Curve of normal distribution is a dumbell shape curve which is symmetrically about mean value




iii) Curve is symmetrically about mean value.

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iv) Since it is symmetrical curve
mean = median = mode

v) x-axis serves as asymptode to normal curve.
asymptode - tangent at Infinity $(\infty)$

vi) Since given curve is symmetric, there is no ~~squeness~~ skewness $\beta_1 = 0$
Kurtosis $\beta_2 = 3$.

Problems :-

1) x is a normal variable and mean of x is 12 and standard deviation is 4 find the following probabilities
i) $x \geq 20$

Sol
i) $x \geq 20$

$\mu = 12$

$\sigma = 4$

$$P(x \geq 20) = \int_{20}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)} dx$$

$$P(x \geq 20) = \int_{20}^{\infty} \frac{1}{4\sqrt{2\pi}} e^{-1/2\left(\frac{x-12}{4}\right)} dx$$

$$= \frac{1}{4\sqrt{2\pi}} \int_{20}^{\infty} e^{-1/2\left(\frac{x-12}{4}\right)} dx$$

at $x = 20$, $z = \frac{x-\mu}{\sigma} = \frac{20-12}{4} = 2$

$$P(x \geq 20) = P(z \geq 2) = \int_{2}^{\infty} \phi(z)\, dz$$

$$= \int_{0}^{\infty} \phi(z)\, dz - \int_{0}^{2} \phi(z)\, dz$$

$$= 1/2 - 0.4772$$

$$= 0.0228$$



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ii) $x \leq 20$

$$P(x \leq 20) = 1 - p(x > 20)$$
$$= 1 - 0.0228$$
$$= 0.772$$

$$P(x \leq 20) = 0.772$$

iii) $P(0 \leq x \leq 12)$

$x = 0$, $z = \frac{x - \mu}{\sigma} = \frac{-12}{4} = -3$

$x = 12$, $z = 0$

$$p(0 \leq x \leq 12) = p(-3 \leq z \leq 0)$$
$$= \int_{-3}^{0} \phi(z)\, dz$$
$$= \int_{0}^{3} \phi(z)\, dz$$
$$= \int_{0}^{3} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$
$$= 0.4987$$



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(v) $P(1.5 < x < 23)$

$x = 1.5$, $Z = \frac{1.5 - 12}{4} = -2.625$

$x = 23$, $Z = \frac{23 - 12}{4} = \frac{11}{4} = 2.75$

$P(1.5 < x < 23) = P(-2.625 < Z < 2.75)$

$$= \int_{-2.625}^{2.75} \phi(z)\, dz$$

$$= \int_{-2.625}^{0} \phi(z)\, dz + \int_{0}^{2.75} \phi(z)\, dz$$

$$= \int_{0}^{2.625} \phi(z)\, dz + \int_{0}^{2.75} \phi(z)\, dz$$

$$= 0.4956 + 0.4970$$

$$= 0.9926$$

* ~~Find the value of x~~

2) let X be a normal variable with mean value 50 and SD = 5 find the following probabilities



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Sol mean = 30 S.D = 5

$\mu = 30 \quad \sigma = 5$

i) $P(26 \le x \le 40)$

$x = 26, \; z = \frac{26-30}{5} = \frac{-4}{5} = -0.8$

$x = 40, \; z = \frac{40-30}{5} = 2$

$P(26 \le x \le 40) = P(-0.8 \le z \le 2)$

$$= \int_{-0.8}^{2} \phi(z)\, dz$$

$$= \int_{-0.8}^{0} \phi(z)\, dz + \int_{0}^{2} \phi(z)\, dz$$

$$= \int_{0}^{0.8} \phi(z)\, dz + \int_{0}^{2} \phi(z)\, dz.$$

$= 0.2881 + 0.4772$

$= 0.7653$



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ii) $P(x \geq 45)$

$x = 45, \quad z = \frac{45-30}{5} = \frac{15}{5} = 3$

$$P(x \geq 45) = P(z \geq 3)$$

$$= \int_{3}^{\infty} \phi(z)\, dz$$

$$= \int_{3}^{\infty} \phi(z)\, dz - \int_{0}^{3} \phi(z)\, dz$$

$$= \int_{0}^{\infty} \phi(z)\, dz - \int_{0}^{3} \phi(z)\, dz$$

$$= \frac{1}{2} - 0.4987$$

$$= 0.0013$$

iii) $|x - 30| > 5$

$\cancel{-(x-30) > 5}$ $\qquad$ $\cancel{x - 30 > 5}$

$\cancel{x - 30 < 5}$ $\qquad$ $x > 35$

$\cancel{x < -5 + 30}$

$x < 25$

$$|x - 30| \leq 5$$

$$= -5 \leq x - 30 \leq 5$$

$$= 25 \leq x \leq 35$$

$$P(25 \leq x \leq 35) = P(-1 < z < +1)$$

$$= \int_{-1}^{1} \phi(z)\, dz = 2\int_{0}^{1} \phi(z)\, dz$$



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$$2\int_0^1 \phi(z)\,dz = 2(0.3413)$$

$$|x-30| > 5 = 1-(|x-30|<5)$$

$$= 1- 2(0.3413)$$

3) The local authority in certain city installed. 10,000 electric lamps in the streets of the city. If these lamps have an avg life of 1000 burning hours with a standard deviation 200 hours. Assuming normality, what no. of lamps might be expected to fail.

i) In first 800 hours

ii) Between 800 to 1200 hours

iii) After what period of burning hours would you expect that 10% of lamps would fail.



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sol 9) $\mu = 1000$ $S.D = 800$

$P(X < 800)$

$x = 800$, $z = \frac{800 - 1000}{800} = \frac{-200}{200} = \frac{-1}{1}$

$= -1$

$P(X < 800) = P(z \le -1)$

$= \int_{-\infty}^{-1} \phi(z)\, dz - \int_{1}^{\infty} \phi(z)\, dz$

$= \int_{0}^{\infty} \phi(z)\, dz - \int_{0}^{1} \phi(z)\, dz$

$= 0.5 - 0.3413$

$= 0.1587$

$10{,}000 \times 0.1587 = 1587$

1587 bulbs are expected to fail in first 800 hours.

ii) $P(800 \le X \le 1200)$

$x = 800$, $z = -1$

$x = 1200$, $z = 1$

$P(800 \le X \le 1200) = P(-1 \le z \le 1)$

$= 2\int_{0}^{1} \phi(z)\, dz$

$= 0.6826$



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$10,000 \times 0.6826$

$= 6826$ bulbs

are expected to fail

b/w 800 to 1200

4) The average ~~mean~~ yeild for 1 acre plot is 662 kilos with a S.D of 32 kilos Assuming normal distribution. How many 1 acre plot in a batch of 1000 plots Would you expect have yeild

i) over 700 kilos

ii) below 650 kilos

iii) lowest yeild of best 100 plots

Sol

$\mu = 662$ kilos S.D $= 32$ kilos

$P(X > 700)$

$X = 700, \ Z = \frac{700 - 662}{32} = \frac{38}{32} = 1.187$

$P(X > 700) = P(Z > 1.187)$

$$= \int_{1.187}^{\infty} \phi(z)\, dz.$$

$$= \int_{0}^{\infty} \phi(z) dz - \int_{0}^{1.187} \phi(z)\, dz$$

$= 0.5 - 0.3810$

$= 0.119$

$N = 1000$

$1000 \times 0.119 = 119$ plots



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$P(X<650)$

$X=650,\ Z=\frac{650-662}{32}=\frac{-12}{32}=-0.375$

$P(X<650)=P(Z<-0.375)$

$=\int_{-\infty}^{-0.375}\phi(z)\,dz$

$=\int_{0.375}^{\infty}\phi(z)\,dz$

$=\int_{0}^{\infty}\phi(z)\,dz-\int_{0}^{0.375}\phi(z)\,dz$

$=0.5-0.1443$

$=0.3557$

$1000\times 0.3557$

$=355$

i) A marks obtained by no. of students for certain subject are assumed to normally distributed with mean value 65 and standard deviation 5. If 3 students are taken at random from list, what is the probability that exactly two of them will have marks more than 70.



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$$z = \frac{70-65}{5} = \frac{5}{5} = 1$$

$$P(x>70) = P(z>1)$$

$$= \int_1^\infty \phi(z)\,dz$$

$$= \int_0^\infty \phi(z)\,dz - \int_0^1 \phi(z)\,dz$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

$n = 3$

$p = 0.1587$

$q = 1-p = 1-0.1587 = 0.8413$

Bi. D

$$P(x=2) = {}^nC_x\, p^x q^{n-x}$$

$$= 3C_2\,(0.1587)^2\,(0.8413)^1$$

$=$

$P(x=2) = 0.9168$



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12/03/2020

Find M.G.F of Normal distribution:-

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = z$

$dx = dz(\sigma)$

$$= \int_{-\infty}^{\infty} e^{t(\mu+z\sigma)} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-1/2 z^2} (\cancel{\sigma}) dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz\sigma} e^{-1/2 z^2} dz$$

$$M_x(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2tz\sigma + t^2\sigma - t^2\sigma^2)} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z-t\sigma)^2} e^{\frac{t^2\sigma^2}{2}} dz$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z-t\sigma)^2} dz$$

Put $z - t\sigma = u$

$$M_x(t) = \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 u^2} du = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$$

$$\because \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 u^2} du = 1$$



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$$\boxed{M_x(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2}}$$

Find the mean & variance of N.D :-

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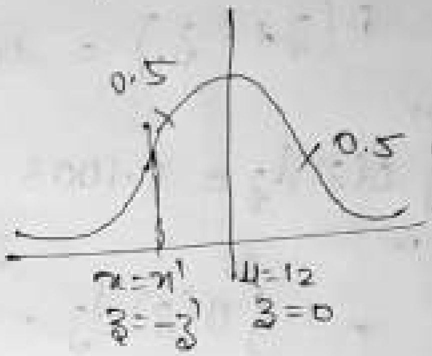
(1) Let x be a Normal variable with mean value 12 & S.D = 4. Find the value of x' if $p(x > x') = 0.24$

Sol Given:- $x = x'$

at $x = x'$, $z = \frac{x-\mu}{\sigma} = \frac{x'-12}{4} = -z_1$

$p(x > x') = p(z > -z_1)$

$= \int_{-z_1}^{\infty} \phi(z)\, dz$



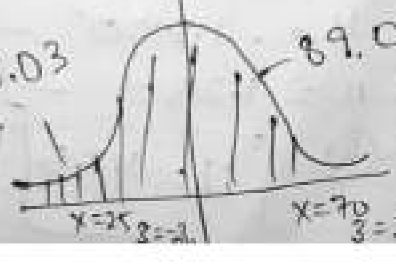

$p(x > x') = p(z > -z_1) = \int_{-z_1}^{\infty} \phi(z)\, dz = 0.24$

$= \int_{-z_1}^{0} \phi(z)\, dz + \int_{0}^{\infty} \phi(z)\, dz = 0.24$

$= \int_{-z}^{0} \phi(z)\, dz = 0.24 - 0.5$

(2) In a distribution exactly normal 10.03% of the items are under 25kg is 89.97% of the items & 70kg what are the mean & S.D?

Sol



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$x = 25$, $z = \frac{x-\mu}{\sigma} = \frac{25-\mu}{\sigma} = -z_1$ —① 

$x = 70$, $z = \frac{x-\mu}{\sigma} = \frac{70-\mu}{\sigma} = z_2$ —②

$P(x < 25) = 10.03\%$

$P(x < 25) = \frac{10.03}{100} = 0.1003$

$P(z < -z_1) = 0.1003$

$$\int_{-\infty}^{-z_1} \phi(z)\, dz = 0.1003 = \int_{z_1}^{\infty} \phi(z)\, dz$$

$$= \int_{0}^{\infty} \phi(z)\, dz - \int_{0}^{z_1} \phi(z)\, dz = 0.1003$$

$$= \int_{0}^{z_1} \phi(z)\, dz = 0.5 - 0.1003$$

$\therefore \boxed{z_1 = 1.28}$

$P(x < 70) = 89.97\%$

$P(z < z_1) = 0.8997 \rightarrow \int_{-\infty}^{z_2} \phi(z)\, dz = 0.8997$

$$\int_{-\infty}^{0} \phi(z)\, dz + \int_{0}^{z_2} \phi(z)\, dz = 0.8992$$

$$\int_{0}^{z_2} \phi(z)\, dz = 0.3997$$

$\boxed{z_2 = 1.28}$

from eq ① $25 - \mu = -1.28\sigma$ — ③

$70 - \mu = 1.28\sigma$ — ④

solve eq ④ & ③ $\boxed{\mu = 47.5}$ $\boxed{\sigma = 17.5}$



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(3) In an examination it is laid down that a student passes if he secures 30% or more marks. He is placed in Ist, IInd & IIIrd Division, According as he gets 60% or marks, Between 45% & 60% & between 30% & 45% respectively. He gets distinction in case he secures 80% or more marks. It is noticed from the result that 10% of the student fails in Examination, where as 5% them obtained distinction. Calculate the % of student placed in second Division.

P(45<x<60)

Sol

$$x = 30, \quad z = \frac{x-\mu}{\sigma} = \frac{30-\mu}{\sigma} = -z_1 \quad \text{--- (1)}$$

$$x = 80, \quad z = \frac{x-\mu}{\sigma} = \frac{80-\mu}{\sigma} = z_2$$

$$P(x<30) = 10\%$$

$$P(z<-z_1) = 0.1$$

$$\int_{-\infty}^{-z_1} \phi(z)\,dz = \int_{z_1}^{\infty} \phi(z)\,dz = 0.1$$

$$= \int_0^{\infty} \phi(z)\,dz - \int_0^{z_1} \phi(z)\,dz = 0.1$$

$$\Rightarrow 0.5 - 0.1 = \int_0^{z_1} \phi(z)\,dz$$

$$= \int_0^{z_1} \phi(z)\,dz = 0.4$$

$$\boxed{z_1 = 1.29}$$



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$P(x \not> 80) = 5\%$ $\frac{5}{100}$

$P(x \not> 80) = 0.05 = P(x < z_2)$

$$\int_{-\infty}^{z_2} \phi(z)\,dz = 0.05$$

$$\int_{-\infty}^{0} \phi(z)\,dz + \int_{0}^{z_2} \phi(z)\,dz = 0.05$$

$$\int_{0}^{z_2} \phi(z)\,dz = 0.05 - 0.5$$

$$= -0.45$$

$$z_2 = 1.65$$

$$\int^{\infty} \phi(z)\,dz = 0.05$$

$$\int_{0}^{\infty} \phi(z)\,dz - \int_{0}^{z_2} \phi(z)\,dz = 0.05$$

$$\int_{0}^{z_2} \phi(z)\,dz = 0.5 - 0.05$$

$$= 0.45$$

$$\boxed{z_2 = 1.65}$$

$\mu - 1.29\sigma = 30$

$\mu + 1.65\sigma = 80$

$30 - \mu = -1.29\sigma$ — (3)

$80 - \mu = 1.65\sigma$ — (4)

$\mu = 51.84$

$\sigma = 17.06$

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$x^2+y^2=a^2$
$x^2+y^2+2xy=0$

# UNIT:- IV

## Fitting of Curves by method of least Squares

→ Fitting of Straight line :-

Consider a data $(x_1, y_1)(x_2, y_2) \ldots (x_n, y_n)$

$$y = ax+b \quad \text{—— (1)}$$

Normal Eqn

$$\Sigma y = a\Sigma x + bn \quad \text{—— (2)}$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x \quad \text{—— (3)}$$

→ Fitting of parabola :- $y = ax^2+bx+c$

Consider a data $(x_1, y_1)(x_2, y_2) \ldots (x_n, y_n)$

$$y = ax^2+bx+c \quad \text{—— (1)}$$

Normal Equations

$$\Sigma y = a\Sigma x^2 + b\Sigma x + Cn \quad \text{—— (2)}$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + C\Sigma x \quad \text{—— (3)}$$

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^2 + C\Sigma x^2 \quad \text{—— (4)}$$



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① Fit a Straight line as well as parabola

| $x$ | $y$ | $xy$ | $x^2$ | $x^3$ | $x^4$ | $x^2y$ |
|---|---|---|---|---|---|---|
| -2 | 1 | -2 | 4 | -8 | 16 | 4 |
| -1 | 5 | -5 | 1 | -1 | 1 | 5 |
| 1 | 13 | 13 | 1 | 1 | 1 | 13 |
| 3 | 21 | 63 | 9 | 27 | 81 | 189 |
| 4 | 25 | 100 | 16 | 64 | 256 | 400 |
| $\Sigma x = 5$ | $\Sigma y = 65$ | 169 | 31 | 83 | 355 | 611 |

$y = ax + b$

N. Eqn $\Sigma y = a\Sigma x + b \times n$

$65 = a(5) + b(5)$

$5b + 5a = 65$

$\boxed{a + b = 13}$ — ①

$\Sigma xy = a\Sigma x^2 + b\Sigma x$

$169 = a(31) + b(5)$

$\boxed{31a + 5b = 169}$ — ②

$a = 4 \quad b = 9$

$\boxed{y = 4x + 9}$



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$y = ax^2 + bx + c$

N. $Eq^n$

$31a + 5b + 5c = 65$

$83a + 31b + 5c = 169$

$355a + 83b + 31c = 611$

$a = 0$

$b = 4$

$c = 9$

$y = 4x + 9$

② 

| x | y |
|---|---|
| -2 | -1 |
| 1 | 4 |
| 4 | 9 |
| 7 | 14 |

| x | y |
|---|---|
| 1 | 1.75 |
| 2 | 7 |
| 3 | 15.75 |
| 4 | 28 |

| x | y |
|---|---|
| 0 | 1 |
| 1 | 0 |
| 2 | 3 |
| 3 | 10 |
| 4 | 21 |

③ fit the following data curve by method of least square

$y = ax^b$ (power curve)

| x | y |
|---|---|
| 1 | 0.5 |
| 2 | 2 |
| 3 | 4.5 |
| 4 | 8 |
| 5 | 12.5 |



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Sol

$$y = a x^b \quad \text{——} \ (1)$$

$$\log_{10} y = \log_{10} a + \log_{10} x^b$$

$$\log_{10} y = \log_{10} a + b \log_{10} x$$

let $\log_{10} y = Y$ $\log_{10} a = A$

$$Y = A + bX \quad \text{——} \ (2)$$

N. eqn

$$\Sigma Y = b \Sigma X + A \cdot n \quad \text{——} \ (3)$$

$$\Sigma XY = b \Sigma X^2 + A \Sigma X \quad \text{——} \ (4)$$

| $x$ | $y$ | $X = \log_{10} x$ | $Y = \log_{10} y$ | $XY$ | $X^2$ |
|---|---|---|---|---|---|
| 1 | 0.5 | 0 | -0.301 | 0 | 0 |
| 2 | 2 | 0.301 | 0.301 | 0.096 | 0.090 |
| 3 | 4.5 | 0.477 | 0.653 | 0.311 | 0.227 |
| 4 | 8 | 0.602 | 0.903 | 0.543 | 0.362 |
| 5 | 12.5 | 0.698 | 1.096 | 0.760 | 0.487 |
| | | 2.078 | 2.646 | 1.704 | 1.166 |

Sub in eq (3) & (4)

(3) $\Rightarrow$ $2.646 = 5A + 2.078b$

(4) $\Rightarrow$ $1.704 = 2.078A + 1.166b$



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$\therefore \quad A = -0.301$

$b = 1.998$

$\log_{10} a = A = -0.301$

$a = (10)^{-0.301}$

$a = 0.501$

$y = 0.501 (x)^{1.998}$



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